Two hours

THE UNIVERSITY OF MANCHESTER

ASYMPTOTIC EXPANSIONS & PERTURBATION METHODS

28 January 2009

09.45 - 11.45

Answer four out of the five questions
A formula sheet is provided on the last page of the exam paper

Electronic calculators may be used, provided that they cannot store text.
1. (a) Show (stating any definitions you use) that, as $x \to 0$

(i) $\sin x = x - \frac{x^3}{3!} + o(x^3)$

(ii) $\cos x = 1 - \frac{x^2}{2!} + O(x^4)$

(iii) $x \sin(1/x) = O(x)$

(6 marks)

(b) Arrange the following functions of $\epsilon$ so that they form an asymptotic sequence as $\epsilon \to 0$, explaining your reasoning

$\epsilon^{-1}, \epsilon \log |\epsilon|, \epsilon^2, \sin \epsilon$, $\epsilon$, $\epsilon^3$

(7 marks)

(c) Using the Fundamental Theorem of Asymptotic Analysis, obtain a two-term asymptotic expansion in powers of $\epsilon$ as $\epsilon \to 0$ for the root of

$x^3 - (3 + \epsilon)x + \epsilon - 2 = 0$

near $x = -1$. Explain what goes wrong if only integer powers of $\epsilon$ are used.

(12 marks)

[TOTAL: 25 marks]

2. (a) Find any critical points of the following system, identify their type and sketch the phase portrait.

\[
\begin{aligned}
\dot{x} &= x + 2x^3 + y \\
\dot{y} &= y + y^3
\end{aligned}
\]

(10 marks)

(b) The equation of motion of the slightly damped pendulum is given by

$\ddot{x} + \epsilon \dot{x} + \sin x = 0$.

Write this ODE as a system of two coupled first order ODEs. By considering of the nature of the system near critical points, for both the case $\epsilon = 0$ and the case $\epsilon > 0$, construct the phase-plane. Explain the effect of $\epsilon$ on the long term behaviour of the system.

(15 marks)

[TOTAL: 25 marks]
3. Let $D$ be a simply connected region of the plane and $\dot{x} = f(x)$ be a plane autonomous system. State and prove a condition on $f$ which guarantees that there is no periodic orbit lying entirely within $D$. (6 marks)

(b) Using (a) what can be said about periodic orbits in the following differential equations?

(i) $\ddot{x} - \dot{x} - x^4 \dot{x} + \sin x = 0$
(ii) $\ddot{x} + (x^2 + \dot{x}^2 - 1) \dot{x} + x = 0$ (12 marks)

(c) For the ODE in (ii) above use the energy balance method to identify a closed curve that nearby solutions tend towards. What result then guarantees that there is a single limit cycle in this case? (7 marks)

[TOTAL: 25 marks]

4. A non-linear oscillator satisfies the equation

$$\ddot{x} + 4x + \epsilon x^3 = 0 \quad \text{with} \quad x(0) = 1, \ \dot{x}(0) = 0$$

(a) Determine the first two terms of an asymptotic expansion for the solution. Explain what is meant by “the expansion is not uniform in $t$”. (12 marks)

(b) Use Lindstedt’s method to obtain a two-term approximation of both the frequency and the position of the oscillator. (13 marks)

[TOTAL: 25 marks]

5. Consider the eigenvalue problem

$$y'' + \frac{\lambda^2}{x^2} y = 0$$

with the boundary conditions

$$y(1) = y(2) = 0.$$ 

(a) Using a trial solution of the form $y = x^\sigma$ show that

$$y = x^{\frac{1}{2}} \left( C \cos \left( \log(x) \sqrt{\lambda^2 - \frac{1}{4}} \right) + D \sin \left( \log(x) \sqrt{\lambda^2 - \frac{1}{4}} \right) \right).$$

Find $C$ and $D$ and hence an expression for the eigenvalues $\lambda_n$ (13 marks)

(b) Use the WKB method to obtain asymptotic approximations to the eigenfunctions, as $\lambda \to \infty$. (12 marks)

[TOTAL: 25 marks]
Formulae

You are given

\[
\cos^3 t = \frac{3}{4} \cos t + \frac{1}{4} \cos 3t
\]
\[
\sin t \cos^2 t = \frac{1}{4} \sin t + \frac{1}{4} \sin 3t
\]
\[
\sin^2 t \cos t = \frac{1}{4} \cos t - \frac{1}{4} \cos 3t
\]
\[
\sin^3 t = \frac{3}{4} \sin t - \frac{1}{4} \sin 3t
\]
\[
2 \cos A \cos B = \cos(A - B) + \cos(A + B)
\]
\[
2 \cos A \sin B = \sin(A + B) + \sin(B - A)
\]
\[
2 \sin A \sin B = \cos(A - B) - \cos(A + B)
\]

The Divergence Theorem or Gauss’ Theorem in the plane states the following: If \( C \) is a closed path that is the boundary of a region \( D \) of the plane and \( (u, v) \) is a vector field that is continuously differentiable on a neighbourhood of \( D \) then

\[
\iint_D \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \, dx \, dy = \oint_C u \, dy - v \, dx
\]