1. Introduction To Perturbation Theory & Asymptotic Expansions

Example 1.0.1. Consider

\[ x - 2 = \varepsilon \cosh x \]  

(1.1)

For \( \varepsilon \neq 0 \) we cannot solve this in closed form. (Note: \( \varepsilon = 0 \Rightarrow x = 2 \))

The equation defines a function \( x : (-\varepsilon, \varepsilon) \to \mathbb{R} \) (some range of \( \varepsilon \) either side of 0)

We might look for a solution of the form \( x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \) and by substituting this into equation (1.1) we have

\[ x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots - 2 = \varepsilon \cosh(x_0 + \varepsilon x_1 + \cdots) \]

Now for \( \varepsilon = 0 \), \( x_0 = 2 \) and so for a suitably small \( \varepsilon \)

\[ x_1 \approx \cosh(2) \]

\[ x(x) = 2 + \varepsilon \cosh(\varepsilon) + \cdots \]

For example, if we set \( \varepsilon = 10^{-2} \) we get \( x = 2.037622 \cdots \) where the exact solution is \( x = 2.039068 \cdots \)

1.1. “Landau” or “Order” Notation.

Definition 1.1.1. Let \( f \) and \( g \) be real functions defined on some open set containing zero, ie: \( 0 \in D \subset \mathbb{R} \). We say:

1. \( f(x) = O(g(x)) \) as \( x \to \infty \) ("Big O")
   
   if \( \exists K > 0 \) and \( \varepsilon > 0 \) such that \( (-\varepsilon, \varepsilon) \in D \) and \( \forall x \in (-\varepsilon, \varepsilon) \left| f(x) \right| < K\left| g(x) \right| \)

2. \( f(x) = o(g(x)) \) as \( x \to \infty \) ("little o") if \( \lim_{x \to 0} \frac{f(x)}{g(x)} = 0 \)

3. \( f(x) \sim g(x) \) as \( x \to \infty \) (asymptotically equivalent) if \( \lim_{x \to 0} \frac{f(x)}{g(x)} = 1 \)

Remark 1.1.2.

1. Could define these for \( x \to x_0 \) or \( x \to \infty \)
2. Abuse of notation: for say, \( \sin x = x + O(x^2) \), \( O(x^2) \) should be an equivalence class, ie: \( \sin x - x \in O(x^2) \)

Lemma 1.1.3. If \( \lim_{x \to 0} \frac{|f(x)|}{|g(x)|} \to m < \infty \) then \( f(x) = O(g(x)) \)

Proof. Suppose

\[ \left| \frac{f(x)}{g(x)} - m \right| < \varepsilon \text{ and } |x| < \delta \]

then

\[ \left| \frac{|f(x)|}{|g(x)|} - |m| \right| < \varepsilon \Rightarrow \frac{|f(x)|}{|g(x)|} < |m| + \varepsilon \]

\[ \Rightarrow |f(x)| < (|m| + \varepsilon)|g(x)| \]

But this is just the definition of \( O \) with \( |m| + \varepsilon \) for \( K \)

Example 1.1.4.

1. \( x^2 = o(x) \) as \( x \to 2 \) since \( \frac{x^2}{x} \to 0 \) as \( x \to \infty \)

2. \( 3x^2 = O(x^2) \) since \( \frac{3x^2}{5x^2} \to 0 \) as \( x \to 0 \)
(iii) \( x = o(|x|^2) \) as \( x \to 0 \)
(iv) \( \frac{2x}{1 + x^2} = o(1) \) since \( \frac{2x/(1 + x^2)}{1} \to 0 \) as \( x \to 0 \)
(v) \( \frac{2x}{1 + x^2} = O(x) \)
(vi) \( \sin x = x + o(x) \) as \( x \to 0 \) since \( \lim_{x \to 0} \frac{\sin x - x}{x} = \lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \)
(vii) \( \sin x = x + O(x^3) \) since \( \lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{x^3} = 0 \)
(viii) \( \sin x \sim x - \frac{x^3}{6} + O(x^4) \)
(ix) \( \sin x = x - \frac{x^3}{6} + O(x^4) \)

The definition of \( f'(x) \) in "o" notation is:
\[
f(x + h) = f(x) + f'(x)h + o(h) \text{ as } h \to 0
\]
The Taylor series for \( f(x + h) \) is given by
\[
f(x + h) = f(x) + f'(x)h + \cdots + \frac{f^{(n)}(x)h^n}{n!} + o(h)
\]
If the Taylor series is a convergent power series then \( o(h^n) \) can be replaced by \( O(h^{n+1}) \).

Any convergent power series \( \sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} a_n x^n + O(x^{n+1}) \)
Examples:
\[
\sqrt{1 + 2x} = 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}(\frac{1}{2})(-2x)^2}{2} + O(x^3) = 1 - x - \frac{x^2}{2} + O(x^3)
\]
\[
\ln(1 + x) = x - \frac{x^2}{2} + O(x^3)
\]
\[
x^2 \sin \left( \frac{1}{x} \right) = O(x^2) \]
It is important to note that this is "big O" with no limit as \( |x^2 \sin \left( \frac{1}{x} \right)| \leq 1 \cdot x^2 \) though \( \frac{|x^2 \sin \left( \frac{1}{x} \right)|}{x^2} = \sin \left( \frac{1}{x} \right) \), and this has no limit.

1.2. The "Fundamental Theorem" of Perturbation Theory.

**Theorem 1.2.1.** If \( A_0 + A_1 \epsilon + \cdots + A_N \epsilon^N = O(\epsilon^{N+1}) \) then \( A_0 = A_1 = \cdots = A_N = 0 \)

**Proof.** Suppose \( A_0 + A_1 \epsilon + \cdots + A_N \epsilon^N = O(\epsilon^{N+1}) \) but not all \( A_k \) are zero. Let \( A_M \) be the first non-zero.

Consider
\[
\frac{A_M \epsilon^M + A_{M+1} \epsilon^{M+1} + \cdots + A_N \epsilon^N}{\epsilon^{N+1}} = \frac{A_M + A_{M+1} \epsilon + \cdots}{\epsilon^{N+1-M}} \to \infty \text{ as } \epsilon \to \infty
\]
Then we have a contradiction with "big O".
1.3. Perturbation Theory of Algebraic Equations.

Example 1.3.1. Consider $x^2 - 3x + 2 + \varepsilon = 0$. Assume the roots have the following expansion: $x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$ then by substitution we get

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 - 3(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \varepsilon + 2 = O(\varepsilon^3)$$

$$(x_0^2 + 3x_0 + 2) + \varepsilon(2x_0 x_1 - 3x_1 + 1) + \varepsilon^2(x_1^2 + 2x_0 x_1 - 3x_2) = O(\varepsilon^3)$$

Terms in $\varepsilon^0$:

$$x_0^2 - 3x_0 + 2 = 0 \Rightarrow x_0 = 1 \text{ or } x_0 = 2$$

Terms in $\varepsilon^1$:

$$2x_0 x_1 - 3x_1 + 1 = 0$$

if $x_0 = 1$ then $x_1 = 1$

otherwise if $x_0 = 2$ then $x_1 = -1$

Terms in $\varepsilon^2$:

$$x_1^2 = 2x_0 x_2 - 3x_2 = 0$$

if $x_0 = 1$ then $x_1 = 1$, and so $x_2 = 1$

otherwise if $x_0 = 2$ then $x_1 = -1$, and so $x_2 = -1$

\[ x = 1 + \varepsilon + \varepsilon^2 + O(\varepsilon^3) \text{ or } x = 2 - \varepsilon - \varepsilon^2 + O(\varepsilon^3) \]

We can solve $x^2 - 3x + 2 + \varepsilon = 0$ directly to get $x = \frac{3 \pm \sqrt{1 - 4\varepsilon}}{2}$

Now $\sqrt{1 - 4\varepsilon} = 1 - 2\varepsilon - 2\varepsilon^2 + O(\varepsilon^3)$ and substituting this into $\frac{3 \pm \sqrt{1 - 4\varepsilon}}{2}$ we get

$$x = \frac{3 \pm (1 - 2\varepsilon - 2\varepsilon^2)}{2}$$

which is the same answer as above.

Example 1.3.2 (Singular Perturbation).

Consider $\varepsilon x^2 - 2x + 1 = 0$ and again assume there is an expansion $x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$. We get for terms in $\varepsilon^0$:

$$-2x_0 + 1 = 0 \Rightarrow x_0 = \frac{1}{2}$$

Terms in $\varepsilon^1$

$$x_0^2 - 2x_1 = 0 \Rightarrow x_1 = \frac{1}{8}$$

Terms in $\varepsilon^2$

$$2x_0 x_1 - x_2 = 6 \Rightarrow x_2 = \frac{1}{8}$$

and so we have

$$x = \frac{1}{2} + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{8}$$

and this gives us one of the roots, but where is the second?

The exact solution is given by:

$$x = \frac{1 \pm \sqrt{1 - \varepsilon}}{\varepsilon}$$

and the other root should be

$$x = \frac{2}{\varepsilon} - \frac{1}{2} + O(\varepsilon)$$
Last time in example 1.3.2, we did not find the other root of \( \varepsilon x^2 - 2x + 1 = 0 \) using the expansion of form \( x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3) \).

If instead we try \( \omega = \varepsilon x \), then \( \frac{d^2}{dt^2} - \frac{1}{\omega^2} + 1 = 0 \Rightarrow \omega^2 - 2\omega + \varepsilon = 0 \) this, we can assume in the usual way has an expansion of the form: \( \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 O(\varepsilon^3) \), and:

Terms in \( \varepsilon^0 \):
\[
\omega_0^2 - 2\omega_0 = 0 \Rightarrow \omega_0 = 0 \text{ or } 2
\]

Terms in \( \varepsilon^1 \):
\[
2\omega_0\omega_1 - 2\omega_1 + 1 = 0 \Rightarrow \omega_1 = -\frac{1}{2} \text{ or } \frac{1}{2}
\]

:. 
\[
\frac{\omega}{\varepsilon} = x = \begin{cases} 
\frac{1}{2} + O(\varepsilon) \\
\frac{1}{2} - \frac{1}{2} + O(\varepsilon)
\end{cases}
\]

**Example 1.3.3** (Non Regular Expansions). Consider \( x^2 - x(2 + \varepsilon) + 1 = 0 \), assume \( x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3) \).

\( \varepsilon^0 : \quad x_0^2 - 2x_0 + 1 = 0 \Rightarrow x_0 = 1 \) twice

\( \varepsilon^1 : \quad (1 + \varepsilon)^2 - (1 + \varepsilon x_1)(2 + \varepsilon) + 1 = O(\varepsilon^2) \Rightarrow 2\varepsilon x_1 - 2\varepsilon x_1 + 1 = O(\varepsilon^2) \Rightarrow 1 = O(\varepsilon^2) \)

This contradicts the assumption that there was a regular expansion.

The exact roots are:
\[
x = \frac{-2 + \sqrt{4 - 4\varepsilon}}{2} = \frac{1}{2}(2 + \varepsilon \pm \sqrt{4 - 4\varepsilon}) = \frac{1}{2}(2 + \varepsilon \pm 2\sqrt{4 - \varepsilon}) = 1 \pm \varepsilon^{\frac{1}{2}} + O(\varepsilon)
\]

Try \( x = x_0 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2 + \cdots \)

\( \varepsilon^0 : \quad x_0 = 1 \) as before.

\( \varepsilon^{\frac{1}{2}} : \quad (1 + \varepsilon^{\frac{1}{2}} x_1)^2 - (1 + \varepsilon^{\frac{1}{2}})(2 + \varepsilon) + 1 = O(\varepsilon^{\frac{1}{2}}) \Rightarrow 2\varepsilon^{\frac{1}{2}} x_1 - 2\varepsilon^{\frac{1}{2}} x_1 = O(\varepsilon) \Rightarrow 0 = 0 \\
\( \varepsilon^1 : \quad (1 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2)^2 - (1 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2)(2 + \varepsilon) + 1 = O(\varepsilon) \)

which gives \( x_1^2 = 1 \Rightarrow x_1 = \pm 1 \) so \( x = 1 \pm \varepsilon^{\frac{1}{2}} + O(\varepsilon^{\frac{1}{2}}) \)

### 1.4. Perturbation Theory of Odes.

**Example 1.4.1** (Regular Problem). Consider the following ODE:

\[ \dot{x} + x = \varepsilon x^2, x(0) = 1 \]

We try an expansion of the form: \( x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2) \) which leads to:

\( \varepsilon^0 : \)
\[
\begin{cases} 
\dot{x}_0 + x_0 = 0 \\
 x_0(0) = 1
\end{cases} \Rightarrow x_0(t) = e^{-t}
\]

\( \varepsilon^1 : \)
\[
\begin{cases} 
\dot{x}_1 + x_1 = x_0^2 = e^{-2t} \\
x_1(0) = 0 \text{ (no } \varepsilon \text{ in } x(0) = 1) \Rightarrow x_1(t) = e^{-t} - e^{-2t}
\end{cases}
\]

and so
\[ x = e^{-t} + \varepsilon(e^{-t} - e^{-2t} + O(\varepsilon^2)) \]
For \( t \in [0, \infty) \) the constants in \( O \) definition can be independent of \( t \).

Sometimes we want only \( \varepsilon > 0 \) versions of \( a, O \) with one sided limits \( \lim_{\varepsilon \to a^+} \).

We use a series \( 1, \varepsilon, \varepsilon^2 \) or in general: \( \varphi_0(\varepsilon), \varphi_1(\varepsilon) \ldots \) \( \varphi_{k+1}(\varepsilon) = o(\varphi_k(\varepsilon)) \) is what is needed.

**Example 1.4.2** (Singular Model Equation).

Consider \( \varepsilon \ddot{x} + x = 1 \), \( x(0) = 0 \), and suppose \( x = x_0 + \varepsilon x_1 + O(\varepsilon^2) \)
Then \( \varepsilon(\dot{x}_0 + \varepsilon x_1) + x_0 + \varepsilon x_1 = 1 + O(\varepsilon^2) \)

\( \varepsilon^0: \ x_0 = 1 \) but \( x(0) = 0 \) cannot solve the initial condition.

We can rescale time, ie: \( t = \varepsilon \tau \) which gives \( \tau = \frac{t}{\varepsilon} \) and \( \frac{dt}{d\tau} = \varepsilon, \frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} = \varepsilon \dot{x} \)

\[
\frac{dx}{d\tau} + x = 1, \ x(0) = 0
\]
Now use \( \varepsilon x = x_0 + \varepsilon x_1 O(\varepsilon^2) \)

\( \varepsilon^0: \)

\[
\frac{dx_0}{d\tau} + x_0 = 1, \ x_0(0) = 0 \Rightarrow x_0 = 1 - e^{-\frac{\tau}{\varepsilon}} = 1 - e^{-\frac{t}{\varepsilon}}
\]

\( \varepsilon^1: \)

\[
\frac{dx_1}{d\tau} + x_1 = 1, \ x_1(0) = 0 \Rightarrow x_1 = 0 \ (\text{similarly for } \varepsilon^2, \varepsilon^3 \text{ etc...})
\]
Therefore the solution is:

\[
x = 1 - e^{-t/\varepsilon}
\]

**Example 1.4.3** ("Singular In The Domain").

Consider \( \dot{x} + \varepsilon x^2 = 1 \), \( x(0) = 0 \), \( t > 0 \) and assume \( x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 O(\varepsilon^2) \)

\( \varepsilon^0: \)

\[
\dot{x}_0 = 1, \ x_0(0) = 0 \Rightarrow x_0 = t
\]

\( \varepsilon^1: \)

\[
(t + \varepsilon x_1) + \varepsilon(t + \varepsilon x_1)^2 = 1 + O(\varepsilon^2) \Rightarrow 1 + \varepsilon \dot{x}_1 + \varepsilon(t^2 + 2tx_1) = 1 + O(\varepsilon^2)
\]

\[
\Rightarrow \varepsilon \dot{x}_1 + \varepsilon t^2 = O(\varepsilon^2) \Rightarrow \dot{x}_1 + t^2 = 0 \Rightarrow x_1 = -\frac{t^3}{3}
\]
(If we carry on we find the \( \varepsilon^2 \) term \( \sim t^5 \)) The solution is:

\[
x = t - \frac{t^3}{3} + O(\varepsilon^3)
\]
This is not regular for \( t \in [0, \infty) \)

**Example 1.4.4** (Damped Harmonic Motion with Small Damping (\( \varepsilon > 0 \))).

Consider \( \ddot{x} + \varepsilon \dot{x} + x = 0 \), \( x(0) = 0 \), \( \dot{x}(0) = 1 \) and assume \( x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 O(\varepsilon^3) \)

\( \varepsilon^0: \)

\[
\ddot{x}_0 + x_0 = 0, \ x_0(0) = 0, \ \dot{x}_0 = 1 \Rightarrow x_0 = \sin t
\]

\( \varepsilon^1: \)

\[
\ddot{x}_1 + \dot{x}_0 + x_1 = 0 \Rightarrow \ddot{x}_1 + x_1 = -\cos t, \ x_1(0) = 0, \ \dot{x}_1(0) = 0
\]
and we should get: \( x_1 = -\frac{1}{2} \varepsilon \sin t \) whereby the solution is:

\[
x = \sin t - \frac{1}{2} \varepsilon \sin t
\]
This again is not uniform for \( t \in [0, \infty) \).
The exact solution is: 
\[
x = e^{-\frac{\epsilon}{2} t} \sin(\sqrt{1 - \frac{\epsilon}{4}} t),
\]
the expansion above is only good for small \( t \).

1.5. Asymptotic Expansions.

**Definition 1.5.1.** A sequence of functions \( \{\varphi_n\}, \ n = 0, 1, 2, \ldots \) is called an asymptotic series as \( x \to x_0 \) if
\[
\lim_{x \to x_0} \frac{\varphi_{n+1}(x)}{\varphi_n(x)} = 0
\]
(i.e: \( \varphi_{n+1}(x) = o(\varphi_n(x)) \))

Note: We could have \( x_0 = \infty \) or a one-sided limit \( x \to x_0^+ \).

Examples:
(i) \( x - \frac{1}{x}, 1, x^2, x, x^3, \ldots \) \( x \to 0^+ \)
(ii) \( 1, \frac{1}{x}, \frac{1}{x\ln(x)}, \frac{1}{x^2\ln(x)}, x \to \infty \)
(iii) \( \tan x, (x - \pi)^2, (\sin x)^3, x \to \pi \)

Taylor’s Theorem:
\[
f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} + R_N(x), \ f \in C^{N+1}[x_0-r, x_0+r], \ r > 0
\]
There are remainder formulas to bound \( R_N(x) \), for example,
Cauchy: \( |R_N(x)| \leq \frac{M_N r^{N+1}}{(N+1)!} \), \( M_N > 0 \), \( R_N = O((x-x_0)^{N+1}) \) as \( x \to x_0 \).

It is important to remember that Taylor \( \Rightarrow R_N \to 0 \) as \( N \to \infty \). In fact we know plenty of power series that do not converge, eg: \( \sum_{n=0}^{\infty} (-1)^n n! x^n \) diverges for all \( x \).

Another famous non-convergent series is Stirling’s formula for \( n! \):
\[
\ln n! = n \ln n + \frac{1}{2} \ln(2\pi n) + \frac{1}{2n^2} \ln n - \frac{1}{2(2^2n)^{2/3}} + O\left(\frac{1}{n^4}\right)
\]

**Definition 1.5.2.** Let \( \{\varphi_n\} \) be an asymptotic sequence as \( x \to x_0 \).
The sum \( \sum_{n=0}^{N} a_n \varphi_n(x) \) is called an asymptotic expansion of \( f \) with \( N \) terms if
\[
f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = o(\varphi_N(x)).
\]
The coefficients \( a_n \) are called the coefficients of the asymptotic expansion. \( \sum_{n=0}^{\infty} a_n \varphi_n(x) \) is called an asymptotic series.

Note: Some people use the stronger definition: \( O(\varphi_N(x)) \)

**Notation 1.5.3.** \( f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x) \) as \( x \to x_0 \)
Clearly any Taylor series is an asymptotic series.

Example: Find an asymptotic expansion for \( f(x) = \int_0^{\infty} \frac{e^{-t}}{1 + xt} \, dt \)

\[(1 + xt)^{-1} = 1 - xt + x^2t^2 + \cdots \implies f(x) = \int_0^{\infty} (1 - xt + x^2t^2 + \cdots)e^{-t} \, dt \]

Now, it can be shown that \( \int_0^{\infty} t^n e^{-t} \, dt = n! \) and hence

\[ f(x) = 1 - x + 2!x^2 - 3!x^3 + \cdots \]

which diverges by the ratio test

\[ \left| \frac{n!x^n}{(n-1)!x^{n-1}} \right| = n|x| \to \infty \forall x \neq 0. \]

It could still be an asymptotic expansion however; we’d need to check

\[ f(x) - \sum_{n=0}^{N} (-1)^n n! x^n = o(x^N) \]

This is a special case of:

**Lemma 1.5.4** (Watson’s Lemma). *Let \( f \) be a function with convergent power series and radius of convergence \( R \), and \( f(t) = O(e^{\alpha t}) \) as \( t \to \infty \) (for some \( \alpha > 0 \)) then:

\[ \int_0^{\infty} e^{-at} f(t) \, dt \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{a^{n+1}} \]

In last example we had \( f(x) = \int_0^{\infty} \frac{e^{-t}}{1 + x} \, dt \) as \( x \to 0^+ \) let \( xt = u, \ a = 1/x \), then

\[ \int_0^{\infty} \frac{e^{-\alpha u}}{u + u^2} \, du \] (looks like Watson’s)

**Example 1.5.5** (Incomplete Gamma Function).

\[ \gamma(a, x) = \int_0^x t^{a-1}e^{-t} \, dt = \int_0^x t^{a-1} dt \sum_{n=0}^{N} \frac{(-t)^n}{n!} \]

\[ = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \int_0^x t^{n+a-1} \, dt = \sum_{n=0}^{N} \frac{(-1)^n}{n!(n+a)} x^{n+a} \]

Note: Power series under integral is convergent, hence uniformly convergent. We have a convergent power series for \( \gamma(a, x) \) in \( x \)

**Example 1.5.6.**

\[ E_i(x) = \int_{-\infty}^x \frac{e^t}{t} \, dt \] (exponential integral)
\[ E_1(x) = \int_1^\infty \frac{e^{-tx}}{t} dt \] (Cauchy principal value), it turns out that \( E_1(x) = -E_i(-x) \)

\[ E_i(x) = \gamma + \ln(x) + x + \frac{x^2}{2} \cdot 2! + \frac{x^3}{3} \cdot 3! + O(x^4) \]

where \( \gamma = \int_0^\infty e^{-x} \ln(x) dx = \lim_{n \to \infty} \left( \sum_{k=0}^{N} \frac{1}{k} - \ln(n) \right) \approx 0.5772 \)

2. ODEs In The Plane

We consider systems of ODEs of the form:

\[
\begin{align*}
\dot{x} &= u(x, y) \\
\dot{y} &= v(x, y)
\end{align*}
\]

where \( x(0) = x_0, y(0) = y_0 \).

(Note: A 2\text{nd} order ODE can be expressed as a coupled system of 1\text{st} order ODEs since if \( \ddot{x} + f(x)\dot{x} + g(x) = 0 \) then if we say \( \dot{x} = y \) it follows that \( \dot{y} = \ddot{x} \) so we get

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -f(x) - g(x)
\end{align*}
\]

Where in this case it turns out that \( u(x, y) = y, v(x, y) = -f(x) - g(x) \))

2.1. Linear Plane autonomous Systems.

\[ \begin{align*}
\dot{x} &= ax + by \\
\dot{y} &= cx + dy
\end{align*} \]

the 1D case is easy: \( \dot{x} = ax, x(t) = x(0)e^{at} \)

Example 2.1.1.

Consider \( \begin{align*}
\dot{x} &= y \\
-3x &= -2y
\end{align*} \)

If we say \( \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \) then we can write \( \ddot{x} = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} \vec{x} \)

We solve these two ODEs to get

\[ x(t) = x(0)e^{-3t}, y(t) = y(0)e^{-2t} \] (which may be written as)

\[ \vec{x}(t) = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \]

We can construct a “Phase plot” with solutions being

curves in the plane called “trajectories” or “orbits”.

We could eliminate \( t \) as follows:

\[ y = y(0) \left( \frac{x}{x(0)} \right)^2, x(0) \neq 0 \]

![Figure 2.1.1]
Example 2.1.2.
Similar to the above example consider \( \begin{align*}
\dot{x} &= -x \\
\dot{y} &= y
\end{align*} \)
We solve in both cases to get \( x(t) = x(0)e^{-t}, y(t) = y(0)e^t \) and eliminate \( t \) such that:
\[
y = y(0) \left( \frac{x}{x(0)} \right)^{-1}
\]
Noting that \( \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \)
we end up with a phase plot which looks like this

![Phase Plot](image)

Figure 2.1.2:

Example 2.1.3 (Simple Harmonic Motion: \( \ddot{x} + \dot{x} = 0 \)).
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x \\
\ddot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]
x(\(t\)) = \( A \cos t + B \sin t \), \( x(0) = A \)
\( \dot{x}(t) = -A \sin t + B \cos t \), \( \dot{x}(0) = B \)
\[
\dot{x} = \begin{bmatrix} x(0) \cos t + y(0) \sin t \\ -x(0) \sin t + y(0) \cos t \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}
\]
The orbits are circles.

![Orbit](image)

Figure 2.1.3:

Example 2.1.4 (Damped Harmonic Motion: \( \ddot{x} + b\dot{x} + x = 0 \)).
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - by \\
\ddot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]
Try \( x(t) = e^{\lambda t} \) giving the characteristic polynomial
\[
\lambda^2 + b\lambda + 1 \Rightarrow \lambda = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - 1}
\]
If \( b \) is small then \( \frac{b^2}{4} - 1 < 0 \) such that

\[
\lambda = \frac{-b}{2} \pm i \sqrt{1 - \frac{b^2}{4}} = \alpha + i\beta
\]

and so

\[
x(t) = e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))
\]

**Figure 2.1.4:**

**Theorem 2.1.5.** Let \( A \subset \mathbb{R}^{2 \times 2} \) be a real matrix with eigenvalues \( \lambda_1, \lambda_2 \) then:

(i) If \( \lambda_1 \neq \lambda_2 \) are real then there exists an invertible matrix \( P \) such that

\[
P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\]

(ii) If \( \lambda_1 = \lambda_2 \) then either \( A \) is diagonal, \( A = \lambda I \) or \( A \) is not diagonal and there is a \( P \) such that

\[
P^{-1}AP = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}
\]

(iii) If \( \lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta, \beta \neq 0 \) then there is a \( P \) such that

\[
P^{-1}AP = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}
\]

How does this help? Put \( \tilde{x} = Ax \) and let \( \tilde{y} = \tilde{P}^{-1}\tilde{x} \)
then \( \tilde{x} = P\tilde{y}, \tilde{y} = \tilde{P}^{-1}\tilde{x}, \tilde{P} \tilde{y} = PA\tilde{x} \)
and so

\[
\tilde{y} = \tilde{P}^{-1}PA\tilde{y}
\]

This allows us to generalise the work we did above.

**Example 2.1.6.** For case 1 in the theorem, consider \( \tilde{u} = \tilde{P}^{-1}AP\tilde{u} \)
then

\[
\tilde{u}_1 = \lambda_1 u_1, \tilde{u}_2 = \lambda_2 u_2
\]

(since \( \tilde{P}^{-1}AP\tilde{u} \) is just a matrix of eigenvectors acting on \( \tilde{u} \))
Then \( u_i(t) = u_i(0)e^{\lambda_i t} \) and so

\[
u(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}
\]

Now \( \tilde{x} = P\tilde{u} \) so \( \tilde{x}(t) = P \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = P \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} P^{-1} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \)

\( u_1 \) and \( u_2 \) are related by eliminating \( t \):

\[
u_1(t) = u_1(0)e^{\lambda_1 t} \Rightarrow \left( \frac{u_1}{u_1(0)} \right)^{\frac{\lambda_1}{\lambda}} = e^t, \text{ then } u_2 = u_2(0)\left( \frac{u_1}{u_1(0)} \right)^{\lambda_2}.
\]
For $\lambda_1 \neq \lambda_2$, distinct eigenvectors $v_1$, $v_2$, $A v_i = \lambda_i v_i$

Half lines through the origin in eigen-directions are trajectories.
2.2. Phase Space Plots.

- **Node: Source**
  - Eigenvalues real, different, same sign
  - Eigenvalues real, different, same sign

- **Node: Sink**
  - Eigenvalues real, different, opposite sign
  - Eigenvalues real, equal, $\lambda > 0, A = \lambda I$

- **Saddle**
  - Eigenvalues real, equal, $\lambda < 0, A = \lambda I$

- **Node: Source**
  - Eigenvalues real, equal, $\lambda < 0, A = \lambda I$
  - Eigenvalues real, equal, $\lambda > 0, A \neq \lambda I$

- **Degenerate Source**
  - Eigenvalues real, equal, $\lambda < 0, A \neq \lambda I$

- **Degenerate Sink**
  - Eigenvalues complex, $\lambda_1 = a + ib, \lambda_2 = a - ib$
  - $\lambda_1 = ib, \lambda_2 = -ib, \beta \neq 0$

- **Stable Spiral**
  - Eigenvalues complex, $\lambda_1 = a + ib, \lambda_2 = a - ib$
  - $\lambda_1 = ib, \lambda_2 = -ib, \beta \neq 0$

- **Unstable Spiral**
  - Eigenvalues complex, $\lambda_1 = a + ib, \lambda_2 = a - ib$
  - $\alpha > 0, \beta \neq 0$

- **Ellipse**

\textbf{Figure 2.2.1:}
Example 2.2.1. consider \(
\begin{align*}
\dot{x} &= -3x + y \\
\dot{y} &= 2x - 2y
\end{align*}
\)

The critical points of this system are at \((0, 0)\), and the Jacobian is given by

\[
\begin{pmatrix}
-3 & 1 \\
2 & -2
\end{pmatrix}
\]

we find the eigenvalues by setting \(A - I = \begin{pmatrix}
-3 - \lambda & 1 \\
2 & -2 - \lambda
\end{pmatrix} = 0\), ie:

\[
\lambda^2 + 5\lambda + 4 = 0 \Rightarrow \lambda = -4, -1
\]

this corresponds to a node (sink)

As for the associated eigenvectors:

\[
A - 4I = \begin{pmatrix}
1 & 1 \\
2 & 2
\end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is in the null space (hence an eigenvector)}
\]

\[
A - I = \begin{pmatrix}
-2 & 1 \\
2 & -1
\end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is the other eigenvector}
\]

We can get further information to help in curve sketching by considering isoclines:

\[
\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{2x - 2y}{-3x + y}
\]

![Figure 2.2.2](image)

2.3. Linear Systems. A linear system for which the eigenvalues are wholly imaginary \((\lambda = \pm \beta)\) is called called a centre.

The characteristic equation, in general of \(A \in \mathbb{R}^{2 \times 2} = \lambda^2 - (\text{trace } A)\lambda + \det A\).

In this case (for \(\lambda\) imaginary), \(\lambda^2 + \beta^2 = 0\), trace \(A = 0\), \(\det A > 0\).

Consider a simple case:

\[
\begin{align*}
\dot{x} &= -y \\
\dot{y} &= cx
\end{align*}
\]

eliminate \(t\) to get

\[
\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{cx}{y}
\]

\[
\Rightarrow \int y \, dy = - \int cx \, dx \Rightarrow \frac{y^2}{2} + \frac{cx^2}{2} = \text{const}
\]

This is the equation for an ellipse.

To determine the direction of the arrows, set \(y = 0\) then on \(x\) axis \(\dot{x} = 0\), \(\dot{y} = cx\)

and \((\dot{x} \; \dot{y})\) is a vector in the direction of solutions, and we that for \(x\) positive \(\dot{x}\) is positive.
The oscillatory nature of these graphs don’t reflect any ‘real life’ situations.

\[ y(t) \]

\[ x(t) \]

**Figure 2.3.1:**

2.4. Linear Approximations.

Consider \( \dot{x} = u(x, y) \), \( \dot{y} = v(x, y) \). Critical points occur when \( u = v = 0 \).

Let \((x_0, y_0)\) be critical points, put \( \xi = x - x_0 \), \( \eta = y - y_0 \) and Taylor expand about \((x_0, y_0)\) to get (near equilibrium point)

\[
\begin{align*}
u(x, y) &= u(x_0, y_0) + \xi \left. \frac{\partial u}{\partial x} \right|_{(x_0, y_0)} + \eta \left. \frac{\partial u}{\partial y} \right|_{(x_0, y_0)} + O(\xi^2 + \eta^2) \quad \text{as} \quad (\xi, \eta) \to (0, 0) \\
u(x, y) &= v(x_0, y_0) + \xi \left. \frac{\partial v}{\partial x} \right|_{(x_0, y_0)} + \eta \left. \frac{\partial v}{\partial y} \right|_{(x_0, y_0)} + O(\xi^2 + \eta^2) \quad \text{as} \quad (\xi, \eta) \to (0, 0)
\end{align*}
\]

and so

\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\eta}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix} + O(\xi^2 + \eta^2)
\]

We can now make the approximation

\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\eta}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}_{(x_0, y_0)}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix} + O(\xi^2 + \eta^2)
\]

which is a linear system.

**Example 2.4.1 (Preditor-Prey).**

We wish to model the dynamics between predators and prey. Without considering external & environmental variables, as the number of predators increases, we expect the population growth rate of the prey should lessen; this then, should result in a slow down in the growth rate of predators (as there is more competition for fewer prey).

Let \( x \) be a population of prey (eg: rabbits), \( y \) be a population of predators (eg: foxes) with \( x, y > 0 \) (note: this model relies on large \( x \) and \( y \) such that we are able to talk about derivatives etc... (since \( x, y \) are integers!))

A simple model is \( \begin{cases} \dot{x} = x(a - \alpha y) \\ \dot{y} = y(-c + \gamma x) \end{cases} \) \((a, c, \alpha, \gamma > 0)\).

Then \( u(x, y) = x(a - \alpha y), \ v(x, y) = y(-c + \gamma x) \) and so for an equilibrium,

\[
\begin{align*}
u = v = 0, \ x(a - \alpha y), \ y(-c + \gamma x) = 0
\end{align*}
\]

so for \( u = 0 \), either \( x = 0 \) or \( a - \alpha y = 0 \) \((y = \frac{a}{\alpha})\)
for \( v = 0 \), either \( y = 0 \) or \(-c + \gamma x = 0 \) \( (x = \frac{c}{\gamma}) \)

Therefore the criticals are at \((0, 0), \left( \frac{c}{\gamma}, \frac{\alpha}{\alpha} \right)\)

More specifically if we put \( a = 1, \alpha = \frac{1}{2}, \ c = \frac{3}{4}, \ \gamma = \frac{1}{4} \) then:

\[
\begin{align*}
\dot{x} &= x(1 - \frac{y}{2}) \\
\dot{y} &= y(-\frac{3}{4} + \frac{x}{4})
\end{align*}
\]

with critical points at \((0, 0), \ (3, 0)\)

Near \((0, 0)\):

\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\eta}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & -\frac{3}{4}
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
\]

The corresponding eigenvalues being:

\[
\lambda_1 = 1, \ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \lambda_2 = -\frac{3}{4}, \ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

This is a saddle. Near \((3, 2)\):

\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\eta}
\end{pmatrix} =
\begin{pmatrix}
0 & -\frac{3}{4} \\
\frac{1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
\]

with eigenvalues:

\[
\lambda_{1,2} = \pm i \frac{\sqrt{3}}{2} \text{ ie: } \xi^2 + \eta^2 = \text{ const, so ellipses.}
\]

Now \[
\frac{dy}{dx} = \frac{y\left(\frac{x}{2} - \frac{3}{4}\right)}{x(1 - \frac{y}{2})} \Rightarrow \frac{3}{4} \ln x + \ln y - \frac{y}{2} - \frac{3}{4} = \text{ const.}
\]

It is possible, albeit tricky to show this is a closed curve.

Figure 2.4.1:

Example 2.4.2 (Circular Pendulum).

We consider a circular pendulum given by \( \ddot{x} + \sin x = 0 \) where \( x \) is an angle.

Figure 2.4.2:

Note that for small angles \( x \), \( x \approx \sin x \) and so we get \( \ddot{x} + x = 0 \) (simple harmonic
We shall solve $\ddot{x} + \sin x = 0$ qualititively since it can’t easily be solved analytically.

Let $\dot{x} = y, \dot{y} = -\sin x = v$

The critical points are at $\dot{x} = \dot{y} = 0 \Rightarrow y = 0, x = n\pi, n \in \mathbb{Z}$

Near the critical point we consider

$$
\begin{pmatrix}
\partial u & \partial u \\
\partial v & \partial v \\
\partial x & \partial y
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-(-1)^n & 0
\end{pmatrix}
at y = 0, x = n\pi
$$

The characteristic equation is

$$
\begin{vmatrix}
-\lambda & 1 \\
(-1)^n & -\lambda
\end{vmatrix} = 0 = \lambda^2 + (-1)^n = 0
$$

If $n$ even $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$, if $n$ odd $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

For $n$ odd we get eigen-vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ which is a saddle.

For $n$ even we get a centre.

- The centres correspond to small swings
- The saddles correspond to swings ‘just’ large enough that they stop at the top
- Everywhere else corresponds to big swings, where with no damping, the pendulum doesn’t stop.

2.5. Non-Linear Operators.

$\ddot{x} + x = 0$ represents simple harmonic motion (SHM) with solution $x(t) = A\cos(t) + B\sin(t)$, a centre.

Consider $\ddot{x} + 2\beta \dot{x} + x = 0$ making the substitution $\dot{x} = y$. The system of ODEs is

given by

$$
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - \beta x^3, u = y, v = -x - \beta x^3
\end{align*}
$$

and so for $0 < \beta < 1$ we get a stable spiral.

Example 2.5.1 (Stiff Spring System).

In a simple spring, force and hence acceleration is proportional to the extension of the spring (Hooke’s Law), instead we think of a stiff spring with force proportional to $x + \beta x^3$. For small $x$ this behaves like $x$, for large $x$, like $x^3$.

$$
\ddot{x} + x + \beta x^3 = 0, \begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - \beta x^3, u = y, v = -x - \beta x^3
\end{align*}
$$

The critical points are at $u = v = 0 \Rightarrow y = 0, x + \beta x^3 = 0 \Rightarrow x = 0$ or $1 + \beta x^2 = 0$ (no real solutions).
The only critical is at $(0, 0)$
\[
\begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 - 3\beta x^2 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]
evaluated at $(0, 0)$
This is a centre $\lambda = \pm i$

**Example 2.5.2** (Soft Spring).
We could change the sign before $\beta$ and simulate a soft spring ie: $\ddot{x} + x - \beta x^3 = 0$
The critical points in this case lie at $(0, 0), (\pm \frac{1}{\sqrt{\beta}}, 0)$ and the jacobian is given by
\[
\begin{pmatrix}
0 & 1 \\
-1 + 3\beta x^2 & 0
\end{pmatrix}
\]
for $x$ close to $\pm \frac{1}{\sqrt{\beta}}$ we have a source; which, in physical terms, means we are actually “adding energy” to the system.

3. **Limit Cycles**
Orbits, that is, trajectories of a system of ODEs cannot cross.

![Figure 3.0.1](image)

If $\vec{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $\vec{x}(0) = \vec{x}(T)$ (for $T \neq 0$) then $\vec{x}(0) = \vec{x}(T)$.
when this happens we have a "periodic orbit", eg a clock, oscillator, cycle in the economy, biology etc...
Supposing we have such a cycle what can be said about what happens around it?
It could be the case that both outside and inside the orbit we spiral towards it; but then again, something else could happen instead. The equilibrium point at the centre doesn’t give us this information.

![Figure 3.0.2](image)

**Example 3.0.3** (Cooked up). Consider a system of the contrived form:
\[
\begin{cases}
\dot{x} = -y - x(x^2 + y^2 - 1) \\
\dot{y} = x - y(x^2 + y^2 - 1)
\end{cases}
\]
we see that by construction, when \( x, y \) satisfy \( x^2 + y^2 = 1 \) we obtain simple harmonic motion. There is only one equilibrium point and that occurs at \((0,0)\). The Jacobian at this point is given by

\[
\begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}_{(x,y)=(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

The characteristic equation is \( \lambda^2 - 2\lambda + 2 = 0 \) which has roots \( \lambda = 1 \pm i \). This is an unstable spiral.

If instead however, we look at this in polar coordinates

\[
r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}
\]

Differentiating implicitly we get:

\[
2r \ddot{r} = 2x \dot{x} + 2y \dot{y}, \quad \dot{\theta} = \frac{xy - y \dot{x}}{x^2 + y^2}
\]

If we multiply (1) and (2) by \( x \) & \( y \) respectively we get;

\[
x \ddot{x} = x(-y - x(x^2 + y^2 - 1)), \quad y \ddot{y} = y(x - y(x^2 + y^2 - 1))
\]

\[
\Rightarrow x \ddot{x} + y \ddot{y} = -(x^2 + y^2)(r^2 - 1) \Rightarrow r \ddot{r} = -r^2(r^2 - 1)
\]

\[
\dot{r} = r(r^2 - 1)
\]

This reveals that there is also an equilibrium point at \( r = 1 \) (ie; for \( r = 1 \) we stay on the circle). Furthermore for \( r > 1 \), \( \dot{r} < 0 \) which is a stable spiral, whilst for \( r < 1 \), \( \dot{r} > 0 \) which is the unstable spiral we found above.

\[\begin{array}{c}
\begin{array}{c}
\theta \\
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
r < 0 \\
r > 0
\end{array}
\end{array}\]

\[\text{Figure 3.0.3:}\]

Now if we multiply (1) and (2) by \( y \) & \( x \) respectively and subtract we get:

\[
x \ddot{y} - y \ddot{x} = x^2 - xy(r^2 - 1) + y^2 + xy(r^2 - 1) = x^2 + y^2 = r^2
\]

\[
\Rightarrow \dot{\theta} = \frac{y^2}{r^2} = 1
\]

The equilibrium point correctly told us that locally we have an unstable spiral but it failed to illuminate the behaviour as we move further out.

We shall establish a couple of results that allow us determine when & where there there exist no closed orbits. First recall

**Theorem 3.0.4** (Divergence Theorem/Divergence theorem in the plane). Let \( C \) be a closed curve, let \( A \subset \mathbb{R}^2 \) be the region it encloses, and \( u, v \) be functions with continuous derivatives. Then

\[
\iint_A \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \, dx \, dy = \int_C u \, dy - v \, dx = \int_C u \frac{dy}{ds} \, ds - v \frac{dx}{ds} \, ds
\]

where \((x(s), y(s))\) is a parameterisation of \( C \)
Theorem 3.0.5 (Bendixon’s Negative criterion). Consider the system \[ \begin{align*}
\dot{x} &= u(x, y) \\
\dot{y} &= v(x, y)
\end{align*} \]
with \(u\) and \(v\) continuously differentiable.

Let \(A \subset \mathbb{R}^2\) be a region of the plane for which \(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\) does not change sign.

Then there is no closed orbit contained within \(A\).

Proof. Suppose for contradiction there exists a closed orbit in \(A\). Then this orbit forms a closed curve \(C\) in \(A\). 

\[ \text{Figure 3.0.4:} \]

Let \(A' \subset A\) be the region enclosed by \(C\) (ie \(\partial A' = C\)). Then by the divergence theorem

\[ \int_0^T (x\dot{y} - y\dot{x}) \, dt = \int_0^T u \frac{dy}{dt} \, dt - v \frac{dx}{dt} \, dt = \int_C u \, dy - v \, dx = \int \int_{A'} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \, dx \, dy = 0 \]

but \(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\) is either \(> 0\) or \(< 0\) \(\forall x, y \in A'\) hence \(\int \int_{A'} \neq 0 \) \(\square\)

Example 3.0.6 (returned to “cooked” example).

\[ \begin{align*}
\dot{x} &= u = -y - x(x^2 + y^2 - 1) \\
\dot{y} &= v = x - y(x^2 + y^2 - 1)
\end{align*} \]

\(\frac{\partial u}{\partial x} = -3x^2 - y^2 + 1, \ \frac{\partial v}{\partial y} = -x^2 - 3y^2 + 1\) and so

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -4x^2 - 4y^2 + 2 = -4(x^2 + y^2) + 2 \]

For values of \(x, y\) such that \(x^2 + y^2 > \frac{1}{2}\), this fails to be always positive or always negative; and so we cannot say there exists no closed orbit. (Though we can be confident there is no such orbit for \(\sqrt{x^2 + y^2} < \frac{1}{\sqrt{2}}\))

Bendixon’s Criterion is not much use for answering the question

Is there a closed orbit between \(r = 1\) or \(r = \frac{1}{\sqrt{2}}\)?

Example 3.0.7 (Damped Harmonic Motion).

For \(\ddot{x} + 2\beta \dot{x} + x = 0\) we have \(\dot{x} = u = y, \ \dot{y} = v = -2\beta y - x\)

This is a stable spiral at \((0,0)\) and

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 - 2\beta \] which is constant

Hence there is no sign change for \(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\) and so no closed orbits.
Example 3.0.8 (General Damped Oscillator).
This system is characterised by $\ddot{x} + f(x)\dot{x} + g(x) = 0$ which with the usual substitution $\dot{x} = y$ yields
\[
\begin{cases}
\dot{x} = y \\
\dot{y} = -f(x)y - g(x)
\end{cases}
\]
Now $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 - f(x)$ is always negative and so general damped systems of this form have no closed orbits.

3.1.

Theorem 3.1.1 (Poincaré-Bendixon Theorem). Given a region $A \subset \mathbb{R}^2$ and an orbit of a system of ODEs $C$ which remains in $A \forall t \geq 0$ then $C$ must approach either a limit cycle or equilibrium.

Remark 3.1.2.

(1) orbit = trajectory = solution curve
(2) limit cycle = closed orbit that nearby orbits approach
(3) We can use this result with time “running backwards”, ie: orbits “come from” unstable equilibrium or closed orbits.

3.2. energy (brief).

Consider an oscillator of the form \[
\begin{cases}
\dot{x} = y \\
\dot{y} = -f(x)
\end{cases}
\]
characterising $\ddot{x} + f(x) = 0$.

Since there is no damping we expect energy to be conserved.

Consider $\varepsilon = \frac{\dot{x}^2}{2} + F(x) = \frac{y^2}{2} + F(x)$ where $\frac{\dot{x}^2}{2}$ can be considered kinetic energy, $F(x)$ the potential energy. Then
\[
\frac{d\varepsilon(x, y)}{dt} = y\dot{y} + f'(x)\dot{x} = (\ddot{x} + F'(x))\dot{x}
\]
Set $F' = f$ then $\frac{d\varepsilon}{dt} = 0$ along solution curves.

So $\varepsilon$ is constant on a solution $(x(t), y(t))$. We call this a “first integral”

Example 3.2.1 (Duffing’s Equation).

This is the “hard spring system” we met earlier $\ddot{x} + \omega^2 x + \varepsilon x^3 = 0$.

$f(x) = \omega^2 x + \varepsilon x^3 \Rightarrow F(x) = \frac{\omega^2 x^2}{2} + \frac{\varepsilon x^4}{4}$ + some constant we need not worry about in this context of constant solution curves. Then $\varepsilon(x, y) = \frac{y^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\varepsilon x^4}{4}$ for $\varepsilon > 0$. As $\varepsilon(x, y)$ is constant then the solutions are bounded for all $t$ (closed curves).

We can say that $x^2 + y^2 \leq \max(2, \frac{2}{\omega^2}) \cdot \left(\frac{y^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\varepsilon x^4}{4}\right) = \max(2, \frac{2}{\omega^2})\varepsilon$. Ie; solutions stay in the circle.

For constant $\varepsilon$ we can check that for $y \neq 0$ there are two solutions for $x$. 
4. Lindstedt’s Method

Example 4.0.2 (Duffing’s Equation). \( \ddot{x} + \omega^2 x + \varepsilon x^3 = 0 \), \( 0 < \varepsilon << 1 \)

We know the solutions are periodic for \( y(0) = \dot{y}(0) \neq 0 \). The solutions will resemble slightly square ellipses.

\[
\ddot{x} + \omega^2 x + \varepsilon x^3 = 0,
\]

\( 0 < \varepsilon << 1 \)

Figure 4.0.1:

consider a straight-forward expansion \( x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 O(\varepsilon^3) \)

Terms in \( \varepsilon^0 \):

\[
\ddot{x}_0 + \omega^2 x_0 = 0 \Rightarrow x_0 = a \cos \omega t
\]

We may suppose without loss of generality that the initial conditions for this system are \( x(0) = a \), and \( \dot{x}(0) = 0 \) since we are just picking out a particular solution curve; we still get all of them.

Terms in \( \varepsilon^1 \):

\[
\ddot{x}_1 + \omega^2 x_1 + x_3^0 = 0 \Rightarrow \ddot{x}_1 + \omega^2 x_1 = -a^3 \cos^3(\omega t)
\]

To deal with \( \cos^3(\omega t) \) we use the identity that

\[
\cos^3(\omega t) = \frac{3 \cos(\omega t)}{4} + \frac{\cos(3\omega t)}{4}
\]

and so

\[
\ddot{x}_1 + \omega^2 x_1 = -a^3 \left( \frac{3 \cos(\omega t)}{4} + \frac{\cos(3\omega t)}{4} \right)
\]

since \( \cos(\omega t) \) appears in the homogeneous solution we would need to introduce a \( \sin(\omega t) \) term (secular term), and this is at odds with what we already know about this system; that it is bounded as \( t \to \infty \).

The trick to get rid of the secular term is to introduce another series:

\[
\tau = \Omega t \text{ where } \Omega = \Omega_0 + \varepsilon \Omega_1 + \cdots
\]

Example 4.0.3 (Lindstedt’s Method). We try again with Duffing’s equation (prefixing \( \varepsilon \) with a minus sign this time) using the above idea and expecting to see a solution that has periodic orbits for \( \varepsilon \) small enough.

Without loss of generality, we rescale time (setting \( \omega = 1 \)) such that we try to solve

\[
\ddot{x} + x - \varepsilon x^3 = 0
\]

We define \( \tau = \Omega t \) where \( \Omega = \Omega_0 + \varepsilon \Omega_1 + \cdots \) and so \( x(t) \) becomes \( x(\tau) \). coupling this with the standard expansion we use for \( x \) we have

\[
x(\tau) = x_0((\Omega_0 + \varepsilon \Omega_1) t) + \varepsilon x_1((\Omega_0 + \varepsilon \Omega_1) t)
\]

We want \( x_i(\tau) = x_i(\tau + 2\pi) \) (i.e: period of \( 2\pi \) for \( i = 1, 2, 3, \ldots \)).

For \( \varepsilon = 0 \) we have \( \Omega_0 = 1 \) (had we not set \( \omega = 1 \) earlier then we’d have instead
\( \Omega_0 = \omega; \) actually we could choose what we want for \( \Omega_0 \) and depending on how
difficult we want to make things, this choice determines \( \Omega_1 \) later. It makes sense
to keep things simple and choose \( \Omega_0 \) such that for \( \varepsilon = 0 \) we have \( x(\tau) = x(\omega \tau) \)
Now \( \frac{d\tau}{dt} = \Omega \) so \( \frac{dx}{dt} = \frac{dx}{d\tau} \frac{d\tau}{dt} = \Omega \frac{dx}{d\tau} \) (so \( \dot{x} = \Omega x' \)) and \( \frac{d^2x}{dt^2} = \Omega^2 \frac{d^2x}{d\tau^2} \)
and so returning to the ODE we have
\[
\Omega^2 x'' - \varepsilon x^3 = 0
\]
\[
\Rightarrow (\varepsilon^{\Omega_0}_1 + \cdots)^2(x''_0 + \varepsilon x''_1 + \cdots) + (x_0 + \varepsilon x_1 + \cdots) - \varepsilon(x_0 + \varepsilon x_1 + \cdots)^3 = 0
\]
Terms in \( \varepsilon^0 \):
\[
x''_0 + x_0 = 0
\]
As before we will assume initial conditions \( x(0) = a, \dot{x}(0) = 0 \) to pick a critical
point on some curve. We know there exists a solution with these properties by
assuming ellipsoidal shaped orbits; and by varying \( a \) we get all the curves.
Since there is no \( \varepsilon \) in initial conditions we get
\[
x_0(0) = a, x_1(0) = 0, x'_0(0) = x'_1(0) = 0
\]
Then the solution for \( x_0 \) is
\[
x_0 = a \cos \tau
\]
Terms in \( \varepsilon^1 \):
\[
x''_1 + 2\Omega_1 x'_0 + x_1 - x_0^3 = 0
\]
\[
\Rightarrow x''_1 + x_1 = -2\Omega_1 (a \cos(\tau)) + (a \cos(\tau))^3 = 0
\]
\[
\Rightarrow x''_1 + x_1 = (2\Omega_1 a + \frac{3}{4} a^3) \cos(\tau) + \frac{1}{4} a^3 \cos(3\tau)
\]
The whole point of doing this was to eliminate the \( \cos(\tau) \) term which would have
introduced a secular term and so we set \( 2\Omega_1 a + \frac{3}{4} a^2 = 0 \)
\[
\Rightarrow \Omega_1 = -\frac{3}{8} a^3
\]
so it now remains to solve
\[
x''_1 + x_1 = \frac{1}{4} a^3 \cos(3\tau)
\]
We try \( x_1 = A \cos(3\tau) \Rightarrow -9A \cos(3\tau) + A \cos(3\tau) = \frac{1}{4} a^3 \cos(3\tau) \Rightarrow A = -\frac{3}{16} a^3 \)
The general solution for \( x_1 \) before applying boundary conditions is
\[
x_1 = \alpha \cos(\tau) + \beta \sin(\tau) - \frac{a^3}{32} \cos(3\tau)
\]
Now \( x_1(0) = 0 \Rightarrow \alpha = -\frac{a^3}{32} \) and \( x'_1(0) = 0 \Rightarrow \beta = 0 \) giving
\[
x_1 = \frac{a^3}{32} \cos(\tau) - \frac{a^3}{32} \cos(3\tau)
\]
Thus
\[
x(t) = a \cos(\Omega t) + \frac{a^3}{32} (\cos(\Omega t) - \cos(3\Omega t))
\]
Where \( \Omega = 1 - \frac{3}{8} a^2 + \cdots \)
5. Method of Multiple scales

In the last section we used Lindstedt’s method to take account of varying frequencies; now we develop a more general method for situations with “two time scales”.

An example of where this is relevant is the damped circular pendulum
\[ \ddot{x} + 2\beta \dot{x} + \sin x = 0 \]

There are two things going on here; there is an oscillation which is captured by one time scale; and a slow loss of energy due to the damping term captured by another time scale. By considering two scales we are able to capture different features of the system.

Figure 5.0.2:

Consider small oscillations and small energy
\[ \ddot{x} + \varepsilon \dot{x} + \sin x = 0 \] (D.H.M)

If we did a standard expansion we’d end up with a solution
\[ x(t, \varepsilon) = \sin t + \frac{\varepsilon}{2} \sin t + \cdots \]

and this is not a uniform expansion for the solution. The exact solution for this system is:
\[ x = \frac{1}{\sqrt{1 - \varepsilon^2}} e^{-\frac{\varepsilon^2}{4}} \sin \left( t \sqrt{1 - \varepsilon^2} \right) \]

where \( x(0) = 1 \) \( \dot{x}(0) = 0 \)

5.1. The Method.

(1) Introduce a new variable \( T = \varepsilon t \), and think of \( t \) as “fast time”, and \( T \) as “slow time”.

(2) Treat \( t \) and \( T \) as “independent variables” for the function \( x(T, t, \varepsilon) \). Using the chain rule we have
\[
\frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial T} \frac{dT}{dt} + \frac{\partial x}{\partial \varepsilon} \frac{\partial \varepsilon}{dt} = \frac{\partial x}{\partial t} + \varepsilon \frac{\partial x}{\partial T}
\]
\[
\frac{d^2x}{dt^2} = \frac{d}{dt} \left( \frac{\partial x}{\partial t} + \varepsilon \frac{\partial x}{\partial T} \right) = \frac{\partial}{\partial t} \left( \frac{\partial x}{\partial t} + \varepsilon \frac{\partial x}{\partial T} \right) + 1 + \frac{\partial}{\partial T} \left( \frac{\partial x}{\partial t} + \varepsilon \frac{\partial x}{\partial T} \right) \varepsilon
\]
\[
= \frac{\partial^2 x}{\partial t^2} + 2\varepsilon \frac{\partial^2 x}{\partial t \partial T} + \varepsilon^2 \frac{\partial x}{\partial T}
\]

(3) Try an expansion \( x = x_0(t, T) + \varepsilon x_1(t, T) + \cdots \) where of course \( T = \varepsilon t \)
(4) Use the extra freedom of $x$ depending on $T$ to kill off any secular terms.

5.2. Application of Multiple scales to D.H.M.

Example 5.2.1. we consider $\ddot{x} + \varepsilon \dot{x} + x = 0$ with boundary conditions $x(0) = 0$, $\dot{x}(0) = 1$, this becomes

$$\frac{\partial^2 x}{\partial t^2} + 2\varepsilon \frac{\partial^2 x}{\partial t \partial T} + \varepsilon^2 \frac{\partial^2 x}{\partial T^2} + \varepsilon \left( \frac{\partial x}{\partial t} + \varepsilon \frac{\partial x}{\partial T} \right) + x = 0$$

Now let $X = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ to get

$$\left( \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial T} + \varepsilon^2 \frac{\partial^2}{\partial T^2} \right) (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \varepsilon \left( \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} \right) (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots = 0$$

Terms in $\varepsilon^0$:

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0$$

this is a PDE with solution

$$x_0 = A_0(T) \cos t + B_0(T) \sin t$$

where $A_0(T), B_0(T)$ are some functions of $T$ (as opposed to just constants) Using boundary conditions $x(0) = 0$, $\dot{x}(0) = 1$ then

$$x_0(0) = A_0(T) \cos 0 + B_0(T) \sin 0 = 0 \Rightarrow A_0(0) = 0$$

$$\dot{x}_0(0) = -A_0(T) \sin 0 + B_0(T) \cos 0 = 1 \Rightarrow B_0(0) = 1$$

Terms in $\varepsilon^1$:

$$\frac{\partial^2 x_1}{\partial t^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial t \partial T} + \frac{\partial x_0}{\partial t} + x_1 = 0$$

Now $\frac{\partial x_0}{\partial t} = A_0(T) \sin t + B_0(T) \cos t$ and $\frac{\partial^2 x_0}{\partial T} = -\frac{dA_0(T)}{dT} \sin t + \frac{dB_0(T)}{dT} \cos t$ and so it remains to solve

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = -2 \left( \frac{dA_0(T)}{dT} \sin t + \frac{dB_0(T)}{dT} \cos t \right) + A_0(T) \sin t - B_0(T) \cos t$$

The RHS contains terms in $\sin(t)$, $\cos(t)$ which will induce secular terms. we therefore choose $A_0(T)$ and $B_0(T)$ such that they “go away”. In other words:

$$2 \frac{dA_0}{dT} + A_0 = 0 \Rightarrow A_0 = A_0(0)e^{-\frac{\varepsilon t}{2}} = 0$$

$$2 \frac{dB_0}{dT} + B_0 = 0 \Rightarrow B_0 = B_0(0)e^{-\frac{\varepsilon t}{2}} = e^{-\frac{\varepsilon t}{2}}$$

and so

$$x_0(t) = e^{-\frac{\varepsilon t}{2}} \sin t = e^{-\frac{\varepsilon t}{2}} \sin t$$

to get the $x_1$ term we would need higher order terms to fully specify it. Notice that with the exception of constants, the first order part captures most of the features of the exact solution.

In general we would have multiple time scales: $T_0 = t$, $T_1 = \varepsilon t$, $\ldots$, $T_n = \varepsilon^n t$

If we consider a series $x_0(T_0, T_1, \ldots, T_n) + \varepsilon x_1(T_0, T_1, \ldots, T_n) + \cdots$

then

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \cdots$$
Example 5.2.2 (Van Der Pol’s Equation). Consider
\[
\dddot{x} + \epsilon (x^2 - 1) \dot{x} + x = 0
\]
Immediately we see that if \( x^2 > 1 \) we have damping, \( x^2 < 1 \) we have negative damping; so it would seem there is a tendency to head towards \( x = 1 \). Perhaps we could use energy methods; anyhow...

\[
\left( \frac{\partial^2}{\partial t^2} + 2i \frac{\partial}{\partial T} + \cdots \right) (x_0 + \epsilon x_1 + \cdots) + \epsilon (x_0^2 + \cdots - 1) \left( \frac{\partial}{\partial \phi} + \epsilon \frac{\partial}{\partial T} \right) (x_0 + \epsilon x_1 + \cdots) + x_0 + \epsilon x_1 + \cdots = 0
\]
Terms in \( \epsilon^0 \):
\[
\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0 \Rightarrow x_0 = A(T) \cos t + B(T) \sin t
\]
We can write this in complex form
\[
x_0 = A_0(T) e^{it} + \overline{A_0(T)} e^{-it} \quad \text{(noting that } z + \overline{z} = 2 \text{Re}(z))
\]
We can justify this step since in general, if \( z = a + ib \) then
\[
(a + ib)e^{it} + (a - ib)e^{-it} = a(e^{it} + e^{-it}) + ib(e^{it} - e^{-it}) = 2a \cos t - 2ib \sin t
\]
Terms in \( \epsilon^1 \):
\[
\frac{\partial^2 x_1}{\partial t^2} + x_1 + 2 \frac{\partial^2 x_0}{\partial t \partial T} + (x_0^2 - 1) \frac{\partial x_0}{\partial t} = 0
\]
\[
\Rightarrow \frac{\partial^2 x_1}{\partial t^2} + x_1 + 2(A' i e^{it} - \overline{A'} e^{-it}) + \left[ (A e^{it} + \overline{A} e^{-it}) - 1 \right] \left[ iA e^{it} - i\overline{A} e^{-it} \right] = 0
\]
where \( A' = A'(T) = \frac{\partial A}{\partial T} \)

If we wish to find \( A(T) \) as opposed to \( x_1 \) we need only consider secular terms. So we need to equate coefficients of \( e^{it} \) (and \( e^{-it} \)) to zero and kill them off.

Considering terms in \( e^{it} \) we have:
\[
2iA' - iA - iA^2 \overline{A} + 2iA \overline{A} = 0
\]
\[
\Rightarrow 2A' - A + A^2 \overline{A} = 0
\]
Similarly, if we consider \( e^{-it} \) we get the conjugate of this expression; ie; whatever \( A \) kills off the \( e^{it} \) terms also kills off the \( e^{-it} \) terms.

We should now use the polar form of \( A \), as \( |A^2 \overline{A}| = |A|^3 \), and if we write \( A = |A|e^{i\varphi} \) with \( \varphi = \text{arg}(A) \) and \( |A| = \frac{1}{2}a \) then
\[
x_0 = a \cos(t + \varphi) \text{ and we now wish to find } a
\]
Now
\[
\frac{dA}{dT} = \frac{1}{2} \frac{da}{dT} e^{i\varphi} + \frac{1}{2} a e^{-i\varphi} \frac{d\varphi}{dT} \text{ and so it remains to solve }
\]
\[
a' e^{i\varphi} + iae^{i\varphi} \varphi' - \frac{1}{2} a e^{i\varphi} + a^2 e^{2i\varphi} \frac{a}{2} e^{-i\varphi} = 0
\]
which dividing through by \( e^{i\varphi} \) gives
\[
a' = \frac{1}{2} a + \frac{1}{2} a^3 - i \varphi' = 0
\]
We now equate real and imaginary parts:
\[
\varphi' = 0 \Rightarrow \varphi \text{ is constant}
\]
Which leaves us the following ODE
\[ a' = \frac{1}{2}a - \frac{1}{8}a^3 \]

There are a number of methods to solve this; one of which being making the substitution \( \cos \theta = \frac{1}{2}a \) and using the fact that
\[
\frac{1}{\sin \theta} = \frac{1}{\sin \theta \csc \theta + \cot \theta} = \frac{\csc^2 \theta + \csc \theta \cot \theta}{\csc \theta + \cot \theta} = -\frac{4}{d} (\csc \theta + \cot \theta)
\]
\[
\Rightarrow \int \frac{1}{\sin \theta} d\theta = -\log(\csc(\theta) + \cot(\theta))
\]

This would still require a bit of work and so we shall instead treat \( a^2 \) as a variable and use partial fractions.

\[
\frac{da^2}{dT} = 2a \frac{da}{dT} = a^2 - \frac{a^4}{4}
\]

\[
\Rightarrow \int \frac{4}{a^2(4-a^2)} da^2 = \int \left( \frac{1}{a^2} + \frac{1}{4-a^2} \right) da^2 = \log|a^2| - \log|4-a^2| = T + T_0
\]

\[
\Rightarrow \frac{a^2}{4-a^2} = ke^T \Rightarrow a^2(1+ke^T) = 4ke^T \Rightarrow a^2 = \frac{4k}{e^T+k}
\]

\[
\Rightarrow a(T) = \frac{2}{\sqrt{1+k^{-1}e^{-T}}}
\]

Hence
\[
x_0(t,T) = \frac{2}{\sqrt{1+k^{-1}e^{-T}}} \cos(t + \varphi)
\]

**Example 5.2.3** (Duffing’s Equation).

We return again to Duffing’s equation \( \ddot{x} + \omega^2 x + \varepsilon x^3 \) in the context of multiple scales. Let \( T = \varepsilon x \) then
\[
\dot{x} = \frac{dx}{dT} = \frac{\partial}{\partial \varepsilon} \frac{\partial}{\partial x}, \quad \ddot{x} = \frac{\partial^2 x}{\partial T^2} = \frac{\partial^2 x}{\partial t^2} + 2\varepsilon \frac{\partial^2 x}{\partial t \partial T} + \varepsilon^2 \frac{\partial^2 x}{\partial T^2}
\]
and so putting this into the ODE above gives
\[
\left( \frac{\partial^2}{\partial T^2} + 2\varepsilon \frac{\partial^2}{\partial T \partial t} \right) (x_0 + \varepsilon x_1 + \cdots) + \omega^2 (x_0 + \varepsilon x_1 + \cdots) + \varepsilon (x_0 + \varepsilon x_1 + \cdots)^3
\]

Terms in \( \varepsilon^0 \):
\[
\frac{\partial^2 x_0}{\partial T^2} + \omega^2 x_0 = 0 \Rightarrow x_0 = Ae^{i\omega t} + A^*e^{-i\omega t}
\]

Terms in \( \varepsilon^1 \):
\[
\frac{\partial^2 x_1}{\partial T^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial T \partial t} + \omega^2 x_1 + x_0^3 = 0
\]
Now
\[
2\frac{\partial^2 x_0}{\partial T \partial t} = 2(A^i\omega e^{i\omega t} - i\omega A^*e^{-i\omega t})
\]
whilst
\[
x_0^3 = A^3 e^{3i\omega t} + 3A^2 Ae^{i\omega t} + 3AA^2 e^{-i\omega t} + A^3 e^{-3i\omega t}
\]
and so
\[
\frac{\partial^2 x_1}{\partial T^2} + \omega^2 x_1 + 2(A^i\omega e^{i\omega t} - i\omega A^*e^{-i\omega t})
\]
\[+ A^3 e^{3i\omega t} + 3A^2 Ae^{i\omega t} + 3AA^2 e^{-i\omega t} + A^3 e^{-3i\omega t} = 0
\]
we now wish to kill off the secular terms $e^{i\omega t}$ and $e^{-i\omega t}$ terms in $e^{i\omega t}$:

$$2i\omega A' + 3A^2 A = 0$$

Again set $A = \frac{1}{2}ae^{i\phi}$ such that $A' = \frac{1}{2}a'e^{i\phi} + \frac{1}{2}aie^{i\phi}\phi'$. Then

$$2i\omega(\frac{1}{2}a'e^{i\phi} + \frac{1}{2}aie^{i\phi}\phi') + \frac{3}{4}a^2 e^{2i\phi} \frac{1}{2}ae^{-i\phi} = 0$$

$$i\omega a' - \omega a\phi' + \frac{3}{8}a^3 = 0$$

The imaginary term $i\omega a' = 0$ \Rightarrow $a = a_0$ (a constant)

This leaves us with a simple ODE to solve

$$\omega a\phi' = \frac{3}{8}a_0^3$$

with solution

$$\phi(T) = \frac{3}{8}a_0^3 \omega \epsilon t \text{ (since } T = \epsilon T)$$

Thus the solution we get working as far as the first term $x_0$ is

$$x = x_0 + O(\epsilon) = a_0 \cos((\omega + \frac{3}{8}a_0^3 \omega)\epsilon t) + O(\epsilon)$$

which as far as the first term goes is identical (except for the sign of $\frac{3}{8}a_0^3 \omega$ (we started with $+\epsilon x^3$ in this example)) to that which we got for Lindstedt’s method.

6. The WKB Method

In this section we focus on eigenvalue problems for 2nd order ODEs, specifically

$$\frac{d^2y}{dx^2} + \lambda^2 r(x)y = 0, \quad y(0) = y(1) = 0 \quad (6.1)$$

ie; $y$ is a function of $x$.

6.1. Motivation. stationary waves, for example, a string on a musical instrument satisfy

$$T \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2}$$

Where $T$ is the tension, $\rho$ the density of string, $y$ the displacement, and $x$ is the distance along the string.

![Figure 6.1.1:](image)

$y(0, t), \ y(1, t) = 0 \Rightarrow$ the string is pinned down at both ends.

Standing wave solutions have the form

$$y(x, t) = y(x) \cos \omega t$$

and so we reduce the wave equation to

$$\frac{d^2y}{dx^2} + \frac{\rho(x)}{T} \omega^2 y = 0$$

this is in the form of (6.1) where $\lambda = \omega$, $r = \frac{\rho(x)}{T}$
6.1.1. **Constant density.** Consider the case \( \rho(x) = \text{constant} \), then

\[
\frac{d^2 y}{dx^2} + \frac{\omega^2}{c^2} y = 0 \quad \text{(where } c^2 = \frac{\sum}{\rho})
\]

This gives

\[
y(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \text{(where } \lambda^2 = \frac{\omega^2}{c^2})
\]

Using the boundary conditions gives

\[
y(0) = 0 \Rightarrow A = 0, \quad y(1) = 0 \Rightarrow B \sin \lambda = 0 \Rightarrow \lambda = \pm n\pi \quad (B \neq 0, \; n \in \mathbb{Z})
\]

thus solutions have the form

\[
y = b \sin(n\pi x) \quad (6.2)
\]

Solutions only exist for \( \lambda \in \pi \mathbb{Z}; \) ie, special or "eigen" values.

In general we only get closed solutions for a simple \( r(x) \).

Note: we are using "boundary conditions" \( y(0), \; y(1) \) rather than initial conditions \( y(0), \; y'(0) \). This means solutions do not exist for all \( \lambda \) (ie; they have to “fit”).

Like “eigenvectors” the eigenfunctions \( y \) can be scaled by any \( B \).

6.2. **The WKB Approximation.** This method stands for *Wentzel, Kramer, Brillouin* (though the work done by Jeffreys 2 years earlier had not been recognised by these three, and so he is often neglected credit for it).

if \( \omega \) or \( \lambda \) are very large, the zeros of (6.2) are very close together. we intend to use an asymptotic expansion for large \( \lambda \).

if \( r \) was constant in (6.1) we would have a solution of the form

\[
y(x) = Ae^{\pm \lambda x \sqrt{r}}
\]

So we will try solutions of the form

\[
y = e^{\lambda g(x, \lambda)}
\]

Then

\[
\frac{dy}{dx} = \lambda g'(\lambda, x)e^{\lambda g(x, \lambda)} \quad \text{and} \quad \frac{d^2 y}{dx^2} = \lambda^2 (g'(\lambda, x))^2 e^{\lambda g(x, \lambda)} + \lambda g''(\lambda, x)e^{\lambda g(x, \lambda)}
\]

we substitute into (6.1) and divide through by \( \lambda^2 e^{\lambda g(\lambda, x)} \) to get

\[
\frac{1}{\lambda} g' + (g')^2 + r = 0
\]

This is a 1st order ODE in \( g' \), ie:

\[
\frac{1}{\lambda} h + h^2 + r = 0
\]

Noting that \( \frac{1}{\lambda} \) behaves like \( \varepsilon \) (since we are considering \( \lambda \) large), we look for an expression of the form

\[
h(x) = h_0(x) + \frac{1}{\lambda} h_1(x) + \frac{1}{\lambda^2} h_2(x) + O(\frac{1}{\lambda^3})
\]

and since \( g = h' = h'' = h'' = h' + \frac{1}{\lambda} h' + \frac{1}{\lambda^2} h' + O(\frac{1}{\lambda^3}) \) then the original expression for \( y \) becomes

\[
y = e^{\lambda g_0 + g_1 + \frac{1}{\lambda} g_2 + O(\frac{1}{\lambda^2})}
\]
The general solution is 

\[ y'' + \lambda^2 xy = 0 \]

using \( y = e^{\lambda g_0 + g_1 + \frac{1}{2}g_2 + O(\frac{1}{x^2})} \) then

\[ y' = e^{\lambda g_0 + g_1 + \frac{1}{2}g_2 + \cdots} (\lambda g_0' + g_1' + \cdots) \]

\[ y'' = e^{\lambda g_0 + g_1 + \frac{1}{2}g_2 + \cdots} (\lambda g_0' + g_1' + \cdots)^2 + e^{\lambda g_0 + g_1 + \frac{1}{2}g_2 + \cdots} (\lambda g_0'' + g_1'' + \cdots) \]

we substitute this into the original expression to get

\[ (\lambda g_0' + g_1' + \cdots)^2 + \lambda g_0'' + g_1'' + \cdots + \lambda^2 x = 0 \]

Terms in \( \lambda^2 \):

\[ g_0'^2 + x = 0 \Rightarrow g_0' = \pm ix^{\frac{1}{2}} \]

\[ \Rightarrow g_0 = \frac{2}{3}ix^{\frac{1}{2}} + A \]

we ignore the constant because as it merely factorises out of \( y \) as a constant multiple of the solution and is of no real value to us.

Terms in \( \lambda^1 \) using \( g_0 = \frac{2}{3}ix^{\frac{1}{2}} \):

\[ g_0'' + 2g_0'g_1' = 0 \Rightarrow \frac{1}{2}ix^{-\frac{1}{2}} + 2ix^{\frac{1}{2}}g_1' = 0 \Rightarrow \frac{1}{4}x + g_1' = 0 \]

\[ \Rightarrow g_1 = -\frac{1}{4} \log(x) + B \]

we didn’t need the absolute value signs for \( \log \) in this case since \( 0 < x, 1 \). Also, if we were to consider the other solution for \( g_0 \) then since \( g_0'' \) inherits the same sign we still get the same solution for \( g_1 \).

The general solution is

\[ y = Ae^{i\lambda \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2} \log(x)} + Be^{-i\lambda \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2} \log(x)} \]

\[ = \frac{1}{\sqrt{x}} \left[ C \cos\left(\frac{2}{3} \lambda x^{\frac{3}{2}}\right) + D \sin\left(\frac{2}{3} \lambda x^{\frac{3}{2}}\right) \right] + O\left(\frac{1}{\lambda^2}\right) \text{ as } \lambda \to \infty \]

In general, if \( y = y'' + \lambda^2 r(x)y = 0 \) then

\[ y(x, \lambda) = \frac{1}{r(x)\sqrt{\lambda}} \left[ A \cos\left(\lambda \int_a^x \sqrt{r(t)} \, dt\right) + B \sin\left(\lambda \int_a^x \sqrt{r(t)} \, dt\right) \right] + O\left(\frac{1}{\lambda^2}\right) \text{ as } \lambda \to \infty \]

Returning to the particular problem, we still haven’t found \( \lambda \) or used boundary conditions.

to find approximate eigenvalues \( \lambda \) for \( \lambda \) large:

\[ y(0) = 0 \Rightarrow C = 0 \]

\[ y(1) = 0 \Rightarrow \sin\left(\frac{4}{3} \lambda\right) = 0 \Rightarrow \frac{4}{3} \lambda = n\pi \]

\[ \Rightarrow \lambda \approx \frac{3n\pi}{4} \text{ (for large } n \in \mathbb{Z}) \]

Example 6.2.2.

this time consider \( y'' + \lambda^2 (1 + 3 \sin^2 x) y = 0, \ y(0) = y(1) = 0 \), again using \( y = e^{\lambda g_0 + g_1 + \cdots} \) such that

\[ (\lambda g_0' + g_1' + \cdots)^2 + (\lambda g_0'' + g_1'' + \cdots) + \lambda^2 (1 + 3 \sin^2 x)^2 = 0 \]

terms in \( \lambda^2 \):

\[ g_0'^2 + (1 + 3 \sin^2 x)^2 = 0 \Rightarrow g_0' = \pm i \sqrt{(1 + 3 \sin^2 x)^2} \]
we now use the angle sum trig identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ to get

$$g_0' = \pm i\left(\frac{5}{2} + \frac{3}{4}(\cos 2x)\right)$$

$$\Rightarrow g_0 = \pm i\left(\frac{5}{2}x - \frac{3}{4}\sin 2x\right)$$

terms in $\lambda^1$ (taking positive $g_0$):

$$2g_0'g_1' + g_0'' = 0 \Rightarrow 2(1 + 3\sin^2 x)g_1' + 3\sin 2x = 0 \Rightarrow g_1' = -\frac{3\sin 2x}{2(1 + 3\sin^2 x)}$$

$$g_1 = -\frac{1}{2}\log(1 + 3\sin^2 x)$$

Again in this example we’d gain no new information if we used the other form of $g_0$. Putting it all together we have

$$y = \frac{1}{(1 + 3\sin^2 x)^{\frac{1}{2}}} \left( C\cos(\lambda\left(\frac{5}{2}x - \frac{3}{4}\sin 2x\right)) + D\sin(\lambda\left(\frac{5}{2}x - \frac{3}{4}\sin 2x\right)) \right)$$

Now we use the boundary conditions

$$y(0) = 0 \Rightarrow C = 0, \quad y(1) = 0 \Rightarrow \lambda\left(\frac{5}{2} - \frac{3}{4}\sin 2\right) = n\pi$$

$$\Rightarrow \lambda_n \approx \frac{n\pi}{\frac{5}{2} - \frac{3}{4}\sin 2}$$

For example, plugging in some numbers, say $n = 8$ then our approximation yields

$$\lambda_8 = 13.824$$

whilst the exact solution is $13.820$.