The aspherical lens formula
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Let z be the coordinate along the axis of a rotationally symmetric lens surface passing through the origin, and r the distance from the axis. If the lens is spherical radius R we see that r and z satisfy

$$(z-R)^2+r^2=R^2$$

or

$$z=R+\sqrt{R^2-r^2}$$

Now consider an ellipse through $(0,0)$ with semi-axes a and b

$$\frac{(z-a)^2}{a^2}+\frac{r^2}{b^2}=1$$

or

$$z=a-a\sqrt{1-\frac{r^2}{b^2}}$$

noting $0\leq z\leq a$. The curvature is

$$\left|\frac{d^2x/dr^2}{(1+(dz/dr)^2)^{3/2}}\right|$$

so the radius of curvature at $r=0$ is

$$R=b^2/a$$

Define the conical constant $p=b/a$ then so $a=R/p^2$ and $b=R/p$ so we can elliminate a and b writing

$$\frac{R^2}{a^2}\left(z-\frac{R}{p^2}\right)^2+\frac{R^2}{b^2}r^2=R^2$$

and hence

$$p^2(z-R/p^2)^2+r^2=\frac{R^2}{p^2}$$

and

$$p^2z^2-2Rz+r^2=0.$$

Now multiply by r^2 and divide by R^2z^2 giving

$$\frac{r^4}{R^2z^2}-\frac{2r^2}{Rz}+\frac{r^2p^2}{R^2}=0$$

which can be written as

$$1-\frac{r^2p^2}{R^2}=\left(\frac{r^2}{Rz}-1\right)^2$$

leading to

$$1+\sqrt{1-\frac{r^2p^2}{R^2}}=\frac{r^2}{Rz}$$

and finally

$$z = \frac{r^2}{R\left(1 + \sqrt{1 - \frac{r^2 p^2}{R^2}}\right)}$$

One advantage of this form is that includes the case of a paraboloid easily by taking $p = 0$ and this is the form often used in the literature[1][2] as the basis of a series expansion for a more general aspherical lenses

$$z = \frac{r^2}{R\left(1 + \sqrt{1 - \frac{r^2 p^2}{R^2}}\right)} + \sum_{j=2}^{\infty} A_j r^{2j}$$

It is also the form used in many optical components catalogues, for example that of Thor Labs.

References

[1] Rudolf Kingslake, Lens design fundamentals p36

[2] Christof Pruss, Eugenio Garbusi, and Wolfgang Osten Testing Aspheres, Optics and Photonics News, Vol. 19, Issue 4, pp. 24-29 (2008) doi:10.1364/OPN.19.4.000024