1. Consider the “negative” heat equation for $u(t, x)$, corresponding to having the thermal diffusivity coefficient $\kappa = -1$:

$$u_t = -u_{xx}.$$ 

Confirm that, for constant values of $A$ and $T$,

$$u = \frac{AT^{1/2}}{(T-t)^{1/2}} \exp\left(-\frac{x^2}{4(T-t)}\right)$$

is a solution for any $t < T$. Use this to show that solutions can exist with, initially, $|u(0, x)| \leq \epsilon$ for any $\epsilon > 0$ but which become infinite in value after any given time later on.

Is the negative heat equation well-posed for $t > 0$ when subjected to initial conditions at $t = 0$?

Can you suggest conditions for which the equation might be well-posed?

2. Use the method of characteristics to find general solutions for the following PDEs for $u(t, x)$ both in terms of a characteristic variable and one of $t$ or $x$, and in terms of $t$ and $x$. In each case sketch the paths of the characteristics.

(a) $u_t - u_x = 0$
(b) $u_t + tu_x = u$
(c) $tu_t - u_x = 1$
(d) $u_t + xu_x = -u$
(e) $xu_t - u_x = t$
(f) $tu_t + xu_x = x$
(g) $tu_t - xu_x = t$
(h) $xu_t - tu_x = xt$
(i) $xu_t + tu_x = -xu$

3. For the general solutions you have obtained from question 2, apply the following boundary conditions, (a) to (a), (b) to (b), etc., and try to obtain unique solutions. For what values of $t$ and $x$ is each solution valid?

(a) $u(0, x) = \cos(x)$
(b) $u(0, x) = \sin(x)$
(c) $u(t, 0) = \exp(-t^2)$
(d) $u(0, x) = x^2$
(e) $u(t, 0) = \ln(1 + t^2)$
(f) $u(1, x) = x^3$
(g) $u(1, x) = 1/(1 + x^2)$
(h) $u(0, x) = 1 + x$ for $x \geq 0$
(i) $u(0, x) = 1 - x$ for $x \geq 0$