Partial Differential Equations — Answer Sheet 3

1. Confirming that \( u = AT^{1/2}(T-t)^{-1/2}e^{-x^2/4(T-t)} \) is a solution of \( u_t = -u_{xx} \):

\[
\begin{align*}
    u_t &= \frac{1}{2}AT^{1/2}(T-t)^{-3/2}e^{-x^2/4(T-t)} - \frac{1}{4}AT^{1/2}x^2(T-t)^{-5/2}e^{-x^2/4(T-t)}, \\
    u_x &= -\frac{1}{2}AT^{1/2}x(T-t)^{-3/2}e^{-x^2/4(T-t)}, \\
    u_{xx} &= -\frac{1}{2}AT^{1/2}(T-t)^{-1}e^{-x^2/4(T-t)} + \frac{1}{4}AT^{1/2}x^2(T-t)^{-5/2}e^{-x^2/4(T-t)} = -u_t
\end{align*}
\]

so that \( u_t = -u_{xx} \).

Note that with this solution \( |u(0, x)| \leq A \) and as \( t \to T \), \( u(t, 0) \to \infty \).

Hence, given any \( \epsilon > 0 \) and any position \((T, X)\), the solution

\[
    u = \epsilon T^{1/2}(T-t)^{-1/2}e^{-(x-X)^2/4(T-t)},
\]

satisfying the initial condition

\[
    u(0, x) = \epsilon e^{-(x-X)^2/4T},
\]

for which \( |u(0, x)| \leq \epsilon \), becomes infinite as \( (t, x) \to (T, X) \).

Because of this, the negative heat equation is ill-posed for \( t > 0 \) when subjected to initial conditions at \( t = 0 \); we can always find solutions that are arbitrarily small initially but that become infinite at any chosen time later on.

The negative heat equation is well posed when subjected to final conditions at some time (say) \( t = t_f \) for times before the final time, \( t < t_f \).

2. (a) \( u_t - u_x = 0 \):

\[
    \frac{dx}{dt} = \frac{du}{T} = \frac{du}{\bar{u}} \quad \text{so}
\]

\[
    \frac{dx}{dt} = -1 \quad \text{and} \quad \frac{du}{dt} = 0
\]

giving, in terms of \((t, k)\):

\[
    x = k - t \quad \text{and} \quad u = A(k).
\]

In terms of \((t, x)\):

\[
    u = A(x + t).
\]

(b) \( u_t + tu_x = u \):

\[
    \frac{dt}{T} = \frac{dx}{t} = \frac{du}{u} \quad \text{so}
\]

\[
    \frac{dt}{dx} = t \quad \text{and} \quad \frac{du}{dt} = u
\]

giving, in terms of \((t, k)\):

\[
    x = k + \frac{1}{2}t^2 \quad \text{and} \quad u = A(k)e^t.
\]

In terms of \((t, x)\):

\[
    u = A(x - \frac{1}{2}t^2)e^t.
\]

(c) \( tu_t - u_x = 1 \):

\[
    \frac{dt}{T} = \frac{dx}{t} = \frac{du}{u} \quad \text{so}
\]

\[
    \frac{dt}{dx} = -t \quad \text{and} \quad \frac{du}{dx} = -1
\]

giving, in terms of \((x, k)\):

\[
    t = ke^{-x} \quad \text{and} \quad u = A(k) - x.
\]

In terms of \((t, x)\):

\[
    u = A(te^{-x}) - x.
\]

(d) \( u_t + xu_x = -u \):

\[
    \frac{dt}{T} = \frac{dx}{x} = \frac{du}{u} \quad \text{so}
\]

\[
    \frac{dx}{dt} = x \quad \text{and} \quad \frac{du}{dx} = -u
\]

giving, in terms of \((t, k)\):

\[
    x = ke^t \quad \text{and} \quad u = A(k)e^{-t}.
\]

In terms of \((t, x)\):

\[
    u = A(xe^{-t})e^{-t}.
\]

(e) \( xu_t - u_x = t \):

\[
    \frac{dt}{T} = \frac{dx}{x} = \frac{du}{u} \quad \text{so}
\]

\[
    \frac{dt}{dx} = -x \quad \text{and} \quad \frac{du}{dx} = -t
\]

giving, \( t = k - \frac{1}{2}x^2 \) and so

\[
    \frac{du}{dx} = -k + \frac{1}{2}x^2.
\]

Hence in terms of \((x, k)\):

\[
    t = k - \frac{1}{2}x^2 \quad \text{and} \quad u = A(k) - kx + \frac{1}{6}x^3.
\]

In terms of \((t, x)\):

\[
    u = A(t + \frac{1}{2}x^2) - (t + \frac{1}{2}x^2)x + \frac{1}{6}x^3.
\]

(f) \( tu_t + xu_x = x \):

\[
    \frac{dt}{T} = \frac{dx}{x} = \frac{du}{u} \quad \text{so}
\]

\[
    \int \frac{dt}{T} = \int \frac{dx}{x} \quad \text{and} \quad \frac{du}{dx} = 1
\]

giving, in terms of \((x, k)\):

\[
    t = kx \quad \text{and} \quad u = A(k) + x.
\]

In terms of \((t, x)\):

\[
    u = A(t/x) + x.
\]
(g) \( tu_t - xu_x = t \); \( \frac{dt}{t} = \frac{dx}{x} = \frac{du}{t} \) so 
\[ \int \frac{dt}{t} = - \int \frac{dx}{x} \text{ and } \frac{du}{dt} = 1 \] giving, in terms 
of \((t, k)\): \( x = k/t \) and \( u = A(k) + t \). 
In terms of \((t, x)\): \( u = A(tx) + t \).

(h) \( xu_t - tu_x = xt \); \( \frac{dt}{x} = \frac{dx}{t} = \frac{du}{tx} \) so 
\[ \int t \, dt = - \int x \, dx \text{ and } \frac{du}{dt} = t \] giving, in terms 
of \((t, k)\): \( x^2 = k - t^2 \) and \( u = A(k) + \frac{5}{2} t^2 \). 
In terms of \((t, x)\): \( u = A(x^2 + t^2) + \frac{1}{3} t^2 \).

(i) \( xu_t + tu_x = -xu \); \( \frac{dt}{x} = \frac{dx}{t} = -\frac{du}{2t} \) so 
\[ \int t \, dt = \int x \, dx \text{ and } \frac{du}{dt} = -u \] giving, in terms 
of \((t, k)\): \( x^2 = k + t^2 \) and \( u = A(k)e^{-t} \). 
In terms of \((t, x)\): \( u = A(x^2 - t^2)e^{-t} \).

3. Note that the solutions below are only valid for those values of \((t, x)\) where 
characteristics passing through \((t, x)\) also pass through the given boundary data without 
either going through infinity or crossing other characteristics. Check this against the 
sketches of the paths of the characteristics . . . .

(a) \( u = A(x + t) \) with \( u(0, x) = \cos(x) \) gives \( A(x) = \cos(x) \). 
Hence \( u = \cos(x + t) \), for all values of \((t, x)\).

(b) \( u = A(x - \frac{1}{2}t^2) e^{-t} \) with \( u(0, x) = \sin(x) \) gives \( A(x) = \sin(x) \). 
Hence \( u = \sin(x - \frac{1}{2}t^2)e^{-t} \), for all values of \((t, x)\).

(c) \( u = A(te^x) - x \) with \( u(t, 0) = \exp(-t^2) \) gives \( A(t) = \exp(-t^2) \). 
Hence \( u = \exp(-t^2e^{2x}) - x \), for all values of \((t, x)\).

(d) \( u = A(xe^{-t})e^{-t} \) with \( u(0, x) = x^2 \) gives \( A(x) = x^2 \). 
Hence \( u = x^2e^{-3t} \), for all values of \((t, x)\).

(e) \( u = A(t + \frac{1}{2}x^2) - (t + \frac{1}{2}x^2)x + \frac{1}{6} x^3 \) with \( u(t, 0) = \ln(1 + t^2) \) gives \( A(t) = \ln(1 + t^2) \). 
Hence \( u = \ln[1 + (t + \frac{1}{2}x^2)^2] - (t + \frac{1}{2}x^2)x + \frac{1}{6} x^3 \), for all values of \((t, x)\).

(f) \( u = A(tx) + x \) with \( u(1, x) = x^3 \) gives \( A(1/x) + x = x^3 \) or \( A(z) = z^{-3} - 1/z \). 
Hence \( u = (x/t)^3 - x/t + x \), for \( t > 0 \).

(g) \( u = A(tx) + t \) with \( u(1, x) = 1/(1 + x^2) \) gives \( A(x) = 1/(1 + x^2) \) or 
\( A(x) = -x^2/(1 + x^2) \). 
Hence \( u = -x^2t^2/(1 + x^2t^2) + t \), for \( t \geq 0 \).

(h) \( u = A(x^2 + t^2) + \frac{1}{2} t^2 \) with \( u(0, x) = 1 + x \) for \( x \geq 0 \) gives \( A(x^2) = 1 + x \) or 
\( A(z) = 1 + \sqrt{z} \). 
Hence \( u = 1 + \sqrt{x^2 + t^2} + \frac{1}{2} t^2 \), for all values of \((t, x)\).

(i) \( u = A(x^2 - t^2)e^{-t} \) with \( u(0, x) = 1 - x \) for \( x \geq 0 \) gives \( A(x^2) = 1 - x \) or 
\( A(z) = 1 - \sqrt{z} \). 
Hence \( u = (1 - \sqrt{x^2 - t^2})e^{-t} \), for \( x \geq |t| \).