Magic Inverse Problems Course - The Radon Transform

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The line $L_{\Theta,s}$ with normal vector $\Theta = (\cos \theta, \sin \theta)$ and distance $s$ to the origin is the set

$$L_{\Theta,s} = \{x \in \mathbb{R}^2 \mid x \cdot \Theta = s\}.$$ 

$\Theta$ takes its values on a circle $S^1 = \{(\cos \theta, \sin \theta) \mid 0 \leq \theta < 2\pi\}$ so the space of lines in $\mathbb{R}^2$ is $S^1 \times (0, \infty)$ (or maybe $S^1 \times \mathbb{R}$ but that counts every line with $s \neq 0$ twice). Topologically this is a cylinder.

We define the Radon transform of a function $f : \mathbb{R}^2 \to \mathbb{R}$ to be a function $R[f](\Theta, s)$ on the set of lines

$$R[f](\Theta, s) = \int_{L_{\Theta,s}} fdl = \int_{x \cdot \Theta = s} fdl.$$ 

We can parameterize a line using the coordinate $t$ as distance along the line. Define $\Theta^\perp = (-\sin \theta, \cos \theta)$, unit vector perpendicular to $\Theta$. Then any $x$ point on $L_{\Theta,s}$ has the form

$$x = s\Theta + t\Theta^\perp,$$ 

in this form

$$\int_{x \cdot \Theta = s} fdl = \int_{t=-\infty}^{\infty} f(s\Theta + t\Theta^\perp) \, dt.$$
Fourier Slice Theorem

We can use the Fourier Transform to better understand the Radon transform. We need the F-T. of a function of 2 variables—or in general $n$-variables

$$\hat{f}(w) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i x \cdot w} dx.$$ 

Some authors miss out the constant, but with this definition, the inverse F-T. is

$$\tilde{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(w) e^{i x \cdot w} dw$$

and of course $\hat{\tilde{f}} = \tilde{\hat{f}} = f$. Also the scaling, translation, differentiation and Perseval’s properties hold from the case $n = 1$. Let us define, for a fixed $\Theta$

$$R_{\Theta}[f](s) = R[f](\Theta, s)$$

for any $s \in \mathbb{R}$. This is all the lines in a particular direction we have.
Theorem (Fourier Slice Theorem)

\[ \hat{R}_\Theta[f](\sigma) = \sqrt{2\pi} \hat{f}(\sigma \Theta). \]

Here \( \sigma \in \mathbb{R} \) is the F-T. (frequency) variable for \( s \).

Proof.

\[
\hat{R}_\Theta[f](\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\Theta \cdot x = s} f(x) e^{-i\sigma s} \, dl \, ds
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{s = -\infty}^{\infty} \int_{t = -\infty}^{\infty} f(s\Theta + t\Theta^\perp) e^{-i\sigma s} \, dt \, ds.
\]

Now

\[
x = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}
\]

which is simply a rotation of the coordinate system \((x, y)\) to \((s, t)\).

\[
\left| \frac{\partial(x, y)}{\partial(s, t)} \right| = 1.
\]

\[
\hat{R}_\Theta[f](\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-i\sigma s} \, dt \, ds
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i\sigma \Theta \cdot x} \, dx \, dy
\]

\[
= \sqrt{2\pi} \hat{f}(\sigma \Theta).
\]
So by taking the Radon transform data, Fourier transformed in $s$, gives us the Fourier transform of the image along each direction. At least where the F.Ts exist in the classical sense this shows that we have uniqueness of solution for the inverse Radon transform.

**Corollary**

*For $f \in S\left(\mathbb{R}^2\right)$, $R[f]$ uniquely determines $f$ (that is $R$ is injective).*

**Proof.**

From FS Theorem $R[f]$ determines $\hat{f}$ on $\mathbb{R}^2$, hence $f$ by the Fourier inversion formula. 

Note, although $\hat{\cdot} : L^2\left(\mathbb{R}^2\right) \rightarrow L^2\left(\mathbb{R}^2\right)$ is invertible, $R$ isn’t necessarily defined on $L^2\left(\mathbb{R}^2\right)$, as as a line is a set of measure zero. You need more smoothness for line integrals to be defined.

It is possible to prove for example that $R$ is defined and continuous, with a continuous inverse in the Sobolev spaces $H^s_0\left(\mathbb{R}^2\right)$ and $H^{s+1/2}(\text{lines})$(see [1] p. 42). (Not quite honest, I should say defined on a bounded subset of $\mathbb{R}^2$).
The Fourier Slice Theorem is not as useful as you might first think. The problem is that if we discretize Θ and σ in regular steps, this produces a non-uniform image resolution.

To derive other reconstruction algorithms we must consider a kind of adjoint.

Recall that for \( K : H_1 \rightarrow H_2, \langle K[f]g \rangle = \langle f, K^*[g] \rangle \) defines the adjoint. For the special case of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), \( K \) could be a matrix and \( K^* \) its transpose.

The 'formal adjoint' of our operator \( K \) defined on functions is the adjoint with respect to the \( L^2 \) norm, even if \( K \) isn't actually defined on \( L^2 \). That is for all \( f \) and \( g \) for which the integrals make sense

\[
\int K[f]g = \int fK^*[g],
\]

where \( K^* \) is the formal adjoint, for example let \( K[f] = f' \) on functions on \( \mathbb{R} \) vanishing outside some interval, then

\[
\int_{-\infty}^{\infty} K[f]g = \int_{-\infty}^{\infty} f'g = [fg]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} fg' = -\int_{-\infty}^{\infty} fK[g].
\]

So \( K^* = -K \), but it isn’t \( K^* \) as it’s not defined on \( L^2(\mathbb{R}) \). The formal adjoint of \( R \), is an operator \( R^* \) which takes Sinogram data, functions on \( S^1 \times \mathbb{R} \) and turns it into ‘images’, functions on \( \mathbb{R}^2 \).

\[
\int_{\mathbb{R}^2} R^*[g](x)f(x) \, dx = \int_{\mathbb{R}} \int_{S^1} R[f](\Theta, s)g(\Theta, s).
\]

So clearly \( R^*[g](x) \) is the integral of \( g \) over all lines through \( x \). It’s called ‘backprojection’ as it throws the data back into the image space. It doesn’t reconstruct however.
In more specific terms $g(\Theta, s)$ backprojected

$$R^*[g](x) = \int_{\theta=0}^{2\pi} g(\Theta, \Theta \cdot x) \, d\theta,$$

where $\Theta = (\cos \theta, \sin \theta)$ as usual, and to make the integral over the $(\Theta, s)$ pairs with $\Theta \cdot x = s$, we have substituted for $s$.

Visually, the effect of $R^* R$ on a small blob is to smear the value of a line integral along a particular line through the blob, all over the line. This turns a blob into a star, at least when infinitely many angles are used.

If one attempted to use ‘backprojection’ as a reconstruction algorithm one would get a ‘starry’ version of the image. This would be the first iteration of Landweber’s method for solving $R[f] = g$, starting with $f_0 = 0$.

Landweber like methods have been used in X-Ray CT where they are called SIRT—simultaneous iterative reconstruction technique, as all the data is ‘backprojected’ at once. By contrast ART—algebraic reconstruction technique ‘backprojects’ one block. e.g., $R_\Theta$ at a time. This is Karczmarz.
Filtered backprojection

This is done in more detail in [1], p. 18–23. We simplify the argument by taking \( n = 2 \).
We define the ‘ramp filter’ operator on a function \( f : \mathbb{R} \to \mathbb{R} \) by

\[
\hat{H}[f](w) = |w| \hat{f}(w)
\]

(It is not standard notation, Natterer calls it \( I^{-1} \)).
Note that it is a bit like a differential operator, indeed \( \hat{H} \circ \hat{H}[f](w) = |w|^2 \hat{f} \), so \( H \circ H = -\partial^2 / \partial \lambda^2 \) the second derivative. So we could legitimately say \( H = \sqrt{-\partial^2 / \partial \lambda^2} \).
Now by definition of the I.F-T.

\[
H[f](x) = (2\pi)^{-1/2} \int_{\mathbb{R}^2} e^{ix \cdot w} |w| \hat{f}(w) \, dw.
\]

(notice here \( f \) is a function of one variable)
Using polar coordinates in the \( w \) plane \( w = \sigma \Theta \) and the Fourier inversion formula.

\[
f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix \cdot w} \hat{f}(w) \, dw
= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \int_{\sigma=0}^{\infty} e^{ix \cdot \sigma \Theta} \hat{f}(\sigma \Theta) \sigma \, d\sigma \, d\theta.
\]
The Fourier Slice Theorem says
\[ \sqrt{2\pi} \hat{f}(\sigma \Theta) = R_{\Theta}[f](\sigma) \]
so
\[
f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\sigma=0}^{\infty} e^{i x \cdot \sigma \Theta} R_{\Theta}[f](\sigma) \sigma \ d\sigma \ d\theta
\]
\[
= \frac{1}{(2\pi)^{3/2}} \int_{\sigma=0}^{\infty} e^{i x \cdot \sigma \Theta} \sigma \hat{g}(\Theta, \sigma) \ d\sigma \ d\theta,
\]
where \( g = R[f] \) and \( \hat{g} \) is the F-T. w.r.t the second variable \( s \).

\[
f(x) = \frac{1}{2} \frac{1}{(2\pi)^{3/2}} \int_{\theta=0}^{2\pi} \int_{\sigma=0}^{\infty} e^{i x \cdot \sigma \Theta} |\sigma| \hat{g}(\Theta, \sigma) \ d\sigma \ d\theta
\]
(remembering the symmetry of \( g \))
\[
= \frac{1}{2} \frac{1}{2\pi} \int_{\theta=0}^{2\pi} H[g](\Theta, \Theta \cdot x) \ d\theta
\]
\[
= \frac{1}{4\pi} R^* H[g].
\]

So in principle we can reconstruct Radon transform data by ‘ramp filtering’ the data then ‘backprojecting’.
More general reconstruction

In the more general case (still $n = 2$) Natterer defines

$$\hat{I}^{\alpha} f(w) = |w|^{-\alpha} \hat{f}(w).$$

So $H = I^{-1}$ which is defined on data (on the cylinder) using the FT wrt the second variable, and on ‘images’ $\mathbb{R}^2 \to \mathbb{R}^2$ using the 2D.FT. He then shows

$$f = \frac{1}{2} (2\pi)^{-1} I^{\alpha} R^* I^{-\alpha} g,$$

where $g = Rf$, which is a reconstruction using a filter on the data and on the backprojected image. For $\alpha > 0$, the image is smoothed (high frequencies suppressed) and high frequencies in the data are sharpened. This type of technique can result in a practical reconstruction algorithm.

(Note that on $S$, $I^\alpha I^{-\alpha}$ is the identity.)

$$f = \frac{1}{2} (2\pi)^{-1} I^{-1} R^* R[f] \quad \alpha = 1.$$ 

So $R^* R[f] = 4\pi I[f]$, from which we can see that the Tikhonov formula

$$f_\mu = \left( R^* R + \mu^2 \right)^{-1} R^* g$$

can be implemented as a filter applied to backprojected data, or conversely

$$f_\mu = R^* \left( R R^* + \mu^2 \right)^{-1} g$$

backprojecting filtered data.
SVD of Radon Transform


We will consider our ‘image’ \( f \) to have support contained in a unit disc centered on the origin \( D \), and \( f \in L^2(D) \).

The codomain of the Radon transform, functions on the cylinder, will need a certain weighted norm.

\[
\| g \|_y^2 = \int_0^{2\pi} \int_{-1}^{1} \frac{|g(s, \Theta)|^2}{\sqrt{1-s^2}} \, ds \, d\theta.
\]

We will call this weighted \( L^2 \) space \( \mathcal{Y} \) and have \( \mathcal{X} = L^2(D), \, R : \mathcal{X} \to \mathcal{Y} \). (Natterer shows that \( R \) and \( R^* \) are both bounded on these spaces.) Let’s check this.

Define \( w(s) = \sqrt{1-s^2} \)

\[
R[f](s, \Theta) = \int_{-w(s)}^{w(s)} f \left( s\Theta + t\Theta^\perp \right) \, dt \quad |s| < 1
\]

\[
|R[f](s, \Theta)|^2 \leq 2w(s) \int_{-w(s)}^{w(s)} \left| f \left( s\Theta + t\Theta^\perp \right) \right|^2 \, dt.
\]

So

\[
\int_{-1}^{1} (w(s))^{-1} |R[f](s, \Theta)|^2 \leq 2 \int_{-1}^{1} \int_{-w}^{w} \left| f \left( s\Theta + t\Theta^\perp \right) \right|^2 \, dt \, ds
\]

\[
= 2 \int_D |f(x)|^2 \, dx
\]

\[
\| g \|_y^2 = \int_0^{2\pi} \int_{-1}^{1} \frac{|g(s, \Theta)|^2}{w(s)} \, ds \, d\theta.
\]

So we have explicitly

\[
\| R[f] \|_y \leq \sqrt{4\pi} \| f \|_x.
\]
Our strategy will be to find eigenfunctions and values of $RR^*$, the adjoint here using the $\mathcal{U}$ norm so it’s not exactly backprojection.

$$R^*[g](x) = \int_0^{2\pi} g(\Theta \cdot x, \Theta)(w(\Theta \cdot x))^{-1} \, d\theta.$$  

And it’s easy to check that $\langle R[f], g \rangle_{\mathcal{U}} = \langle f, R^*[g] \rangle_{\mathcal{X}}$ for any $f \in \mathcal{X}$, $g \in \mathcal{U}$.

We will need some ‘special functions’. Special functions arise mainly as the eigenfunctions of self-adjoint ordinary differential equations. They are orthogonal bases for weighted $L^2$ spaces on intervals. They are usually associated with 19th century European mathematicians, and their properties are tabulated in great detail in old textbooks and the help system of ‘Mathematica’.

Natterer uses Gegenbauer polynomials $C_\lambda^\ell$, which are orthogonal on $[-1, 1]$ with weight $(1 - x^2)^{\lambda-1/2}$. So

$$\int_{-1}^{1} (1 - x^2)^{\lambda-1/2} C_\lambda^\ell(x) C_\lambda^k(x) \, dx = \begin{cases} 0 & \ell \neq k \\ \text{horrible nonzero formula} & \ell = k \end{cases}$$

For $\lambda = 0$ these are ‘Chebyshev polynomials of the first kind’ $T_\ell = C_0^\ell$ which have a nice simple formula

$$T_\ell(x) = \cos \left( \ell \cos^{-1} x \right)$$

It is a polynomial!
And $\lambda = 1$ Chebychev polynomials of the second kind
\[ U_\ell(x) = (\ell + 1) C_\ell^1 = \frac{\sin(\ell + 1) \cos^{-1} x}{\sin \cos^{-1} x} \quad \ell = 0, 1, 2, \ldots \]

The orthogonality result is
\[ \int_{-1}^{1} w(x) U_m(x) U_m'(x) \, dx = \frac{\pi}{2} \delta_{m,m'} . \]

By a change of variables $x = \cos \theta \quad 0 < \theta < \pi$
\[ U_m(\cos \theta) = \frac{\sin(m + 1) \theta}{\sin \theta} \]

and
\[ \int_{-1}^{1} w(x) U_m(x) U_m'(x) \, dx = \int_{0}^{\pi} \sin(m + 1) \theta \sin(m' + 1) \theta \, d\theta = \frac{\pi}{2} \delta_{m,m'} . \]

Now if $g(s)$ is in the $w^{-1}$ weighted $L^2$ space
\[ \|g\|^2 = \int_{-1}^{1} \frac{|g(s)|^2}{w(s)} \, ds = \int_{0}^{\pi} |g(\cos \theta)|^2 \, d\theta . \]

And $g$ has an expansion in these orthogonal function
\[ g(\cos \theta) = \sum_{m=0}^{\infty} c_m \sin(m + 1) \theta \]
\[ c_m = \frac{2}{\pi} \int_{0}^{1} g(\cos \theta) \sin(m + 1) \theta \, d\theta . \]

So we can represent $g(s)$ as a series expansion in $u_m(s) = w(s) U_m(s)$. 
Now we do a trick rather like ‘separation of variables’ for solving Laplace’s equation.
We think of $g(s, \Theta)$ as having a series in $w(s)U_m(s)$ for each fixed $\Theta$.
We are lead to consider the subspaces of $\mathcal{Y}$ of functions of the form
$$\sqrt{2/\pi}w(s)U_m(s)u(\theta)$$
for any $u \in L^2(0, 2\pi)$ call this $\mathcal{Y}_m$.
We have $\mathcal{Y} = \mathcal{Y}_0 \oplus \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \cdots$ and it is clear that each subspace $\mathcal{Y}_m$
is orthogonal to the others.
Our important fact is that $RR^*\mathcal{Y}_m \subseteq \mathcal{Y}_m$, so that we can look for
eigenfunctions of $RR^*$ in each $\mathcal{Y}_m$. In matrix terms $RR^*$ is
‘block-diagonal’.
Now we can show with a little calculation (It's not bad, remark 4.4 on p. 243 of [2]) that any \( g_m \in \mathcal{Y}_m \), 
\( g_m(s, \Theta) = \sqrt{2/\pi} w(s) U_m(s) u(\Theta) \) satisfies

\[
RR^*[g_m](s, \Theta) = \sqrt{\frac{2}{\pi}} \int_{-w(s)}^{w(s)} U_m \left( \Theta' \cdot (s \Theta + t \Theta^\perp) \right) u(\Theta') \, d\theta' \, dt
\]

\[
= \frac{4\pi}{m+1} \sqrt{\frac{2}{\pi}} w(s) U_m(s) \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin(m+1)(\theta - \theta')}{\sin(\theta - \theta')} u(\theta') \, d\theta'
\]

which shows that \( RR^* \mathcal{Y}_m \subseteq \mathcal{Y}_m \) as claimed.

Now we have to understand

\[
u(\Theta) \mapsto \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin(m+1)(\theta - \theta')}{\sin(\theta - \theta')} u(\theta') \, d\theta
\]

You may have met this integral operator before in the context of Fourier series the operator taking \( u \) to its
truncated Fourier series (the Fejér kernel)

Specifically let \( y_\ell(\theta) = 1/\sqrt{2\pi} e^{i\ell \theta} \) be a Fourier basis function (orthonormal on \([0, 2\pi]\)), then

\[
\frac{1}{2\pi} \frac{\sin(m+1)(\theta - \theta')}{\sin(\theta - \theta')} = \sum_{k=0}^{m} y_{m-2k}(\theta) \overline{y}_{m-2k}(\theta').
\]

With this in mind let us define

\[
u_{m,k}(s, \Theta) = \frac{2}{\sqrt{\pi}} w(s) U_m(s) y_{m-k}(\theta) \quad k = 0, 1, \ldots, m
\]

which are orthonormal in \( y_m \).

We now have

\[
RR^* u_{m,k} = \sigma_m^2 u_{m,k} \quad k = 0, 1, \ldots, m.
\]

\[
\sigma_m = \left( \frac{4\pi}{m+1} \right)^{1/2}
\]

We can now find \( v_{m,k} = (1/\sigma_m)R^* u_{m,k} \) with a little more 'special function technology'.

It turns out that
\[ \nu_{m,k}(x) = (2m + 2)^{1/2} Q_{m,|m-2k|}(|x|) y_{m-2k} \left( \frac{x}{|x|} \right), \]

where \( Q_{m,|m-2k|} = r^\ell P^{(0,\ell)}_{1/2(m+\ell)}(2r^2 - 1) \) where \( P \) is a Jacobi polynomial (a what?)

The Fourier Slice Theorem shows us that the \( u_{m,k} \) form a complete orthonormal system on \( Y \) so we have the SVD of the Radon transform.

In particular it tells us the Radon transform is only mildly ill-posed. The singular values \( \sigma_m \) are each associated with \( m + 1 \) singular functions \( u_{m,0}, u_{m,1}, \ldots, u_{m,m} \), so counting multiplicity the singular values are

\[ \sigma_0 = \sqrt{4\pi}, \sigma_1 = \sqrt{\frac{4\pi}{2}}, \sqrt{\frac{4\pi}{2}}, \sigma_2 = \sqrt{\frac{4\pi}{3}}, \sqrt{\frac{4\pi}{3}}, \sqrt{\frac{4\pi}{3}}, \ldots, \]

so the singular values are decaying more slowly than \( 1/\sqrt{m+1} \).