This EBL really is much more “enquiry based”. We have n’t done 2nd order ODEs
yet in the lecture so you get to try some things yourself.

A constant coefficient 2nd order linear ODE is of the form

\[ a \frac{d^2y}{dt^2} + b \frac{dy}{dt} + cy = f \]

where \( y(t) \) and \( f(t) \) are functions of \( t \) and \( a \neq 0 \). We will concentrate on \( f = 0 \). There
are lots of example in HELM 19.3

1. Consider the ODE

\[ \frac{d^2y}{dt^2} + \omega^2 y = 0 \]

where \( \omega \) is a non zero constant.

(a) Check that \( y(t) = \sin \omega t \) and \( y(t) = \cos \omega t \) both satisfy the ODE.
(b) Check that \( y(t) = A \sin \omega t + B \cos \omega t \) is a solution for any \( A \) and \( B \).

2. Consider the ODE

\[ \frac{d^2y}{dt^2} - k^2 y = 0 \]

where \( k \) is a non zero constant.

(a) Check that \( y(t) = e^{kt} \) and \( y(t) = e^{-kt} \) both satisfy the ODE.
(b) Check that \( y(t) = Ae^{kt} + Be^{-kt} \) is a solution for any \( A \) and \( B \).

3. Consider the ODE

\[ \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 0 \]

(a) Try a solution of the form \( y = e^{kt} \). Show that for it to be a solution we must
have \( k^2 - k - 6 = 0 \)
(b) Solve the quadratic and write down two solutions of the ODE.

The equation for \( k \) is called the auxiliary equation and \( y = e^{kt} \) is called a trial
solution

4. Find the auxiliary equation for the following ODEs and find the solutions \( k \) for
these auxiliary equations. Note if there are one or two solutions and if they are
real or complex.

(a)

\[ \frac{d^2y}{dt^2} - 9y = 0 \]

(b)

\[ \frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0 \]
(c) \[
\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0
\]

(d) \[
\frac{d^2y}{dt^2} + 9y = 0
\]

5. Considering 4(d) above the auxiliary equation tells us the solutions are $e^{3it}$ and $e^{-3it}$. Using the fact that $e^{i\theta} = \cos \theta + i\sin \theta$ show that

\[e^{3it} + e^{-3it} = 2\cos 3t\]

and

\[e^{3it} - e^{-3it} = 2i\sin 3t.\]

In general if we have a solution $y = Ce^{\beta t} + De^{-\beta t}$ then $y = A\cos \beta t + B\sin \beta t$ where $A = (C + D)$ and $B = (C - D)/i$ is also a solution.

6. In 4(c) we only get one solution $e^{-t}$. Check that in this case $te^{-t}$ is also a solution.

In general if the auxiliary equation has complex roots $k = \alpha \pm i\beta$ then the real solutions are of the form $y(t) = e^{\alpha t}(A\cos \beta t + B\sin \beta t)$

7. (a) Check that the auxiliary equation of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 4y = 0$ has roots $k = -1\pm \sqrt{3}i$.

(b) The general solution is going to be $y(t) = e^{-t}(A\cos \sqrt{3}t + B\sin \sqrt{3}t)$ check that the special case $y(t) = e^{-t}\cos \sqrt{3}t$ is actually a solution.