Answer all parts of Section A (40 marks in all) and one of the two questions in Section B (10 marks)

Electronic calculators may be used, provided that they CANNOT store text.

General Instruction. All numerical working must be shown.
SECTION A

Answer ALL questions

A1. Given

\[ A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} \]

calculate \( \det A \) and \( A^{-1} \) [2 marks]

A2. Let

\[ A = \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}. \]

Compute the following matrices:

(a) \( A + 2B \). Is it true that \( A + 2B = 2B + A \)?

(b) \( A^T + B^T \). Is it true that \( (A + B)^T = A^T + B^T \)?

(c) \( AB \) and \( BA \). Is it true that \( AB = BA \)?

(d) \( B^T A^T \). Is it true that \( (AB)^T = B^T A^T \)? [8 marks]

A3. Evaluate the following 3 \( \times \) 3 determinants:

\[ d_1 = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & -1 \\ -1 & -2 & 2 \end{vmatrix}, \quad d_2 = \begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 3 \\ -1 & 2 & -2 \end{vmatrix}. \]

How are \( d_1 \) and \( d_2 \) related and why? [Hint: use the fact that \( \det(AB) = \det(A) \times \det(B) \).] [4 marks]

A4. Given

\[ x + y + z = 3 \]
\[ x - y + 2z = 2 \]
\[ y - z = 5. \]

(a) Write the system of equations in the form of \( Ax = b \).

(b) Calculate the determinant of \( A \).

(c) Solve for \( x, y, z \) using Gaussian elimination without pivoting. [6 marks]
A5. Solve analytically the following differential equation problem: find \( y(x) \) such that
\[
\frac{dy}{dx} = x\sqrt{1 - y^2}; \quad y = 0 \text{ when } x = 0.
\]

[Hint: use the fact that \( \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x \).]

[4 marks]

A6. Solve the following problem by finding an integrating factor: find \( y(x) \) such that
\[
\frac{dy}{dx} + \frac{y}{x} = \sin x; \quad y = 0 \text{ when } x = \pi.
\]

[5 marks]

A7. Compute a numerical solution to the differential equation problem
\[
\frac{dy}{dt} = -y + t + 1; \quad y = 1 \text{ when } x = 0,
\]
by taking four steps of the forward Euler method with step size \( h = 0.1 \).

[5 marks]

A8. Write down Newton’s iteration for finding the roots of a function \( f(x) \). Hence or otherwise, determine an iterative method to calculate the cube root of a given real number, \( \beta \) say. Starting with an initial guess of \( x_0 = 2 \), run the iteration and calculate \( \sqrt[3]{10} \) correct to four decimal places.

[6 marks]
SECTION B

Answer ONE of the two questions

B9. Let \( A \) represent a general \( k \times k \) matrix.

(a) Explain as succinctly as possible how you might use Gaussian Elimination to find the inverse of the matrix \( A \).  

\[ \text{[4 marks]} \]

(b) Apply the technique to compute the inverse of the following matrix

\[ A = \begin{pmatrix} -2 & 2 \\ 1 & 2 \end{pmatrix}. \]

\[ \text{[3 marks]} \]

(c) Describe the process of Gaussian Elimination with partial pivoting and give the main reason why including pivoting is important when solving linear equation systems using a computer.

\[ \text{[3 marks]} \]

B10. Explain what is meant by the terminology of a nonlinear first-order differential equation problem

\[ \frac{dy}{dt} = f(t, y) \]

and give an example of such an equation.

\[ \text{[2 marks]} \]

(a) Given some initial condition and a step length of \( h \), write down the general update formula for computing the implicit Euler solution \( y_{n+1}^* \) from the previous estimate \( y_n^* \approx y(nh) \). Explain carefully how you would compute \( y_{n+1}^* \) in practice. Give one reason why the implicit Euler method might be used in preference to the forward Euler method, when solving a nonlinear differential equation problem.

\[ \text{[5 marks]} \]

(b) Take a step size of \( h = 0.1 \) and compute the first two steps of the implicit Euler method applied to the linear problem

\[ \frac{dy}{dt} = -y + t + 1; \quad y = 1 \text{ when } x = 0. \]

\[ \text{[3 marks]} \]