1. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of full rank and $b \in \mathbb{R}^m$ be a vector
   (a) i. Assuming $m \geq n$ derive a formula for
   
   $$x_{MP} = \arg \min_{x \in \mathbb{R}^n} ||Ax - b||^2$$

   and explain why the minimum is unique.

   ii. Now assuming $n \geq m$ we define

   $$x_{MP} = A^T (AA^T)^{-1} b.$$  

   Show that this satisfies $Ax_{MP} = b$.

   Show that for any $v \in \text{null}(A)$, $||x_{MP} + v|| \geq ||x_{MP}||$. What minimization problem does $x_{MP}$ solve?

   (b) Again taking $m \geq n$, an iteration scheme is defined by

   $$x_0 = 0$$

   $$x_{k+1} = x_k + \tau A^T (b - Ax_k)$$

   i. Show by induction that

   $$x_k = \left( I - (I - \tau A^T A)^k \right) (A^T A)^{-1} A^T b$$

   ii. Given the singular value decomposition $A = U \Sigma V^T$ find a diagonal matrix $D_k$ such that

   $$x_k = VD_k U^T b$$

   and hence show that for $0 < \tau < 2/\sigma_1^2$, $x_k \to x_{MP}$ as $k \to \infty$, where $\sigma_1$ is the largest singular value of $A$. 

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2. (a) i. State three conditions which are generally taken to characterise a well-posed problem

ii. Let $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a linear operator between (infinite dimensional) Hilbert spaces, define the terms *bounded* and *compact* as applied to $K$. Show that the inverse of a compact operator is not bounded.

iii. Let $K : L^2(0, 1) \rightarrow L^2(0, 1)$ be the integral operator

$$K[g](x) = \int_0^1 k(x, y) f(y) \, dy.$$  

State conditions on the kernel function $k$ which guarantee that $K$ is compact.

(b) An integral operator $K : L^2(0, 1) \rightarrow L^2(0, 1)$ has kernel function

$$k(x, y) = \begin{cases} 
  x(y - 1) & x < y \\
  y(x - 1) & x \geq y 
\end{cases}$$

i. Given $f(0) = f(1) = 0$ show that for $f \in C^2(0, 1)$, $K[f''] = f$.

ii. Find the adjoint $K^*$ and verify that $u_m(x) = -\sqrt{2} \sin(m\pi x)$ is a right singular function of $K$, giving the left singular functions and singular values. Hence or otherwise show that the operator $K$ is compact.

iii. Is the inverse problem of determination of $f \in L^2(0, 1)$ from $g = K[f] \in L^2(0, 1)$ moderately or severely ill-posed? What choice of Hilbert space for the domain would make the inverse problem well posed?
3. (a) Let $A \in \mathbb{R}^{m \times n}$ be of full rank, $b \in \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$, $\alpha > 0$ and

$$\tilde{A} = \begin{pmatrix} A \\ \alpha I \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b \\ \alpha x_0 \end{pmatrix}.$$  

Show that $x_T = \tilde{A}^\dagger \tilde{b}$ is the unique minimizer of

$$||Ax - b||^2 + \alpha^2 ||x - x_0||^2$$

stating carefully any properties of the Moore-Penrose generalized inverse you use.

(b) Show that an alternative formula for the minimizer in (i) is

$$x_T = x_0 + A^T (AA^T + \alpha^2 I)^{-1} (b - Ax_0).$$

(c) Use a suitable substitution to find a formula for the Tikhonov regularised solution using more general norms:

$$x_{GT} = \arg \min_{x \in \mathbb{R}^n} ||Ax - b||_Q^2 + \alpha^2 ||x - x_0||_P^2$$

where $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{m \times m}$ are positive definite symmetric matrices and $||x||_P^2 = x^T P x$.

(d) In a practical problem you are required to solve $Ax = b$ where $A$ is ill-conditioned and $b$ contaminated by experimental error. Discuss briefly strategies which might be used to select $P$, $Q$, $x_0$ and $\alpha$ in the generalized Tikhonov regularised solution (your answer should be no more than half a page).
4. (a) i. Define the Fourier transform $\hat{f}$ and convolution $f \ast g$ where $f$ and $g$ are suitable functions on $\mathbb{R}$.

ii. Let

\[ g(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \leq 1/2 \end{cases} \]

Find the Fourier transform of $g$, and the convolution $g \ast g$.
Find the Fourier transform of the convolution $\hat{g} \ast \hat{g}$ stating any results you use.

(b) The Radon transform of a function $f$ on $\mathbb{R}^2$ is defined as the line integral

\[ R[f](\Theta, s) = R_{\Theta}[f](s) = \int_{\Theta \cdot x = s} f(x) \, dx \]

where $\Theta \in S^1$ is a unit vector in $\mathbb{R}^2$ and $s \in \mathbb{R}$.

i. Show that

\[ \hat{R}_\Theta[f](\sigma) = c \hat{f}(\sigma \Theta) \]

for a constant $c$.

ii. Find the formal adjoint $R^*[g](x)$ of $R$ applied to a function $g$ on $S^1 \times \mathbb{R}$ and evaluated at a point $x \in \mathbb{R}^2$

iii. Define the term back-projection as used by the computerized tomography community.
Given that

\[ (RR^* + \alpha^2)^{-1}[g] = h_\alpha \ast g \]

for some function $h_\alpha$, justify the assertion that Tikhonov regularization of the inverse Radon transform is equivalent to back-projecting filtered data.
Is there any mathematical justification for using simply the back-projection of the data as a reconstruction algorithm?