1.  
   i) Define the terms *fixed point* and *periodic point* (of a dynamical system).
    Define the *prime period* of a periodic point.

   ii) Let \( S \) be a set with a finite number of elements, and let \( f: S \to S \) be a mapping.
    Show that \( S \) contains at least one periodic orbit (which could be a fixed point).
    Hence (or otherwise) show that if there is no nonempty subset \( T \subset S \) (except \( S \) itself) such that \( fT = T \), then \( S \) consists solely of a single periodic orbit.

   iii) Let \( f: X \to X \) be a mapping of an arbitrary set \( X \), and let \( x \) be a periodic point
    of \( f \). Let \( p_1, p_2, \ldots \) be the periods of \( x \), with \( p_1 < p_2 < \ldots \). Show that \( p_2, p_3, \ldots \) are all multiples of \( p_1 \).

   iv) Let \( S_n \) be the set of strings of 0’s and 1’s of length \( n \). (Thus \( S_3 \) contains
    000, 001, 011, etc.). Let \( f: S_n \to S_n \) be the mapping that moves the leftmost
    element of a string to its right hand end. (Thus \( f(001) = 010, f(101) = 011 \)
    etc.).
    Show that \( f: S_n \to S_n \) has precisely two fixed points whatever the value of \( n \).
    Show that for any \( s \in S_n \), \( s \) is a period \( n \) point of \( S_n \). For what values of \( n \)
    does \( f: S_n \to S_n \) have periodic points (other than fixed points) whose prime
    period is not \( n \)? (Justify your answer.)
2.  
  
  i) Let \( f: \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) function, and let \( x_s \) be a fixed point of \( f \). Say what it means for \( x_s \) to be a hyperbolic fixed point. Show that if \( |f'(x_s)| < 1 \) then there is an interval \( U = (x_s - \varepsilon, x_s + \varepsilon) \) such that, for every \( x_0 \in U \),

\[
\left| \frac{x_{n+1} - x_s}{x_n - x_s} \right| \leq A < 1
\]

for all \( n = 0, 1, \ldots \). (Here, as usual, \( x_n = f^n(x_0) \), and \( A \) is a constant.)

ii) A superstable periodic orbit is one that contains a critical (i.e. stationary) point of \( f \). Show that if \( x_s \) is a superstable fixed point then there is an interval \( U \), as above, such that, for every \( x_0 \in U \),

\[
\left| \frac{x_{n+1} - x_s}{x_n - x_s} \right| \to 0
\]

for all \( n = 0, 1, \ldots \).

iii) For the logistic map \( F_\mu(x) = \mu x(1 - x) \)

(a) Find the value of \( \mu \) for which the nonzero fixed point \( p_\mu \) is superstable.

(b) By considering the position of the critical point of \( F_\mu \) show that the value of \( \mu \) for which the period 2 orbit is superstable satisfies the equation \( \mu^3 - 4\mu^2 + 8 = 0 \). Prove that this equation has a single real root in the range \( 3 \leq \mu \leq 1 + \sqrt{6} \).
3. Consider the parameterized family of maps \( f_\lambda : \mathbb{R} \rightarrow \mathbb{R} \) where \( f_\lambda(x) = x^3 - \lambda x \)

\( i) \) Sketch graphs of \( f_\lambda \) for \( \lambda < -1, \lambda = 0 \) and \( \lambda > 1 \).

\( ii) \) Find all the fixed points of \( f_\lambda \) and the values of \( \lambda \) for which each exists.

Sketch a bifurcation diagram for a small interval of \( \lambda \) around \( \lambda = -1 \), indicating both the stable and unstable fixed points (and making clear which are which). What kind of bifurcation takes place at \( \lambda = -1 \)?

\( iii) \) At what value of \( \lambda \) does the fixed point at 0 undergo a period doubling bifurcation?

Show that if \( x = p \) is a period 2 point of \( f_\lambda \) then so is \( x = -p \). Assuming these together form a period 2 orbit, find \( p \) as a function of \( \lambda \). Show that \( p \) does not undergo any further period doubling bifurcations, however large \( \lambda \) is made.

\( iv) \) Now consider the family \( f_\lambda(x) = x^3 - \lambda x + \varepsilon \) where \( \varepsilon \) is a small positive constant. Sketch a bifurcation diagram around \( \lambda = -1 \), showing how it differs from that in \( ii) \). (You do not need to give a rigorous justification of this sketch.) What new bifurcation has appeared?
4. Consider the logistic map \( F_\mu : \mathbb{R} \rightarrow \mathbb{R} \), \( F_\mu (x) = \mu x (1 - x) \), where \( \mu > 4 \).

   i) Sketch the graph of \( F_\mu \).

   ii) Some points in the interval \([0, 1]\) have orbits that eventually leave the interval. If \( x \) is such a point, what happens to \( F_\mu^n(x) \) as \( n \rightarrow \infty \)? Illustrate this using graphical analysis.

   iii) Show that the set of \( x \in [0, 1] \) such that \( F_\mu(x) \in [0, 1] \) consists of two intervals (say \( I_0 \) and \( I_1 \)). Find the intervals \( I_0 \) and \( I_1 \) (i.e. find their end points).

   iv) Show that there is a value of \( \mu \), say \( \mu_* \), such that if \( \mu > \mu_* \) then \( |F'_\mu(x)| > 1 \) for all \( x \in I_0 \cup I_1 \). What is the smallest value we can take for \( \mu_* \)?

   v) Let \( \Lambda \) be the set of points in \([0, 1]\) which never leave it under iteration of \( F_\mu \) (i.e. \( \Lambda = \{ x \mid F_\mu^n(x) \in [0, 1], n = 0, 1, 2, \ldots \} \)). Let \( \Sigma_2 \) be the sequence space of two symbols (0 and 1). Define the map \( h : \Lambda \rightarrow \Sigma_2 \), \( h(x) = s_1 s_2 s_3 \ldots \), by saying \( s_i = 0 \) if \( F_\mu^i(x) \in I_0 \) and \( s_i = 1 \) if \( F_\mu^i(x) \in I_1 \). If \( \mu > \mu_* \) show using iv) (or otherwise) that \( h \) is injective.