1. Consider the autonomous differential equation

\[
\frac{dx}{dt} = F(x) \tag{1}
\]

where \( x \in \mathbb{R}^n \). Assume that for any \( x_0 \in \mathbb{R}^n \) and \( t \in \mathbb{R} \) there is a unique solution passing through \( x_0 \) at \( t_0 \), and the solution exists for all time.

\( i \) What is meant by ‘autonomous’?

\( ii \) Show that if \( x(t) \) is a solution then so is \( y(t) = x(t + \tau) \) for any (fixed) \( \tau \in \mathbb{R} \). Write down a condition that \( y(t) \) satisfies.

\( iii \) Give a definition of the time \( t \) map, \( \phi_t \), of the system (where \( t \in \mathbb{R} \)). What is the flow of the system?

Use the result of \( ii \) to show that \( \phi_s \circ \phi_t = \phi_{t+s} \). What function is \( \phi_0 \)?

\( iv \) The differential equation \( \frac{dx}{dt} = (a - x)x \), where \( x \in \mathbb{R} \) and \( a \) is a constant, with initial condition \( x(0) = x_0 \), has the solution

\[
x(t) = \frac{ax_0e^{at}}{a-x_0(1-e^{at})}
\]

Write down the time \( t \) map and use the formula to show that \( \phi_s \circ \phi_t = \phi_{t+s} \).

\( v \) Show that if equation (1) is linear (i.e. \( F(x + y) = F(x) + F(y) \) and \( F(ax) = aF(x) \) for all \( x, y \in \mathbb{R}^n \) and constant \( a \in \mathbb{R} \)) then \( \phi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is linear.
2. \( i \) Let \( f: \mathbb{R} \to \mathbb{R} \) and let \( p \) be a fixed point of \( f \). Define asymptotic stability of \( p \). Show that if \( f \) is \( C^1 \) then \( |f'(p)| < 1 \) implies that the fixed point \( p \) is asymptotically stable.

\( ii \) Define \( f_\lambda: [0, \infty) \to [0, \infty) \) by \( f_\lambda(x) = \lambda x e^{-x} \), where \( \lambda > 0 \).

\( a \) Sketch a graph of this function.

\( b \) Show 0 is a fixed point of \( f_\lambda \) for all \( \lambda \). Show that for \( \lambda > 1 \) there is another fixed point, \( p_\lambda \), and find an expression for \( p_\lambda \) in terms of \( \lambda \). Show that for \( \lambda = 3/2 \), \( p_\lambda \) is asymptotically stable.

\( c \) As \( \lambda \) is increased from 0, \( p_\lambda \) undergoes a period doubling bifurcation. Find the value of \( \lambda \) at which this occurs. Find the value of \( p_\lambda \) when the bifurcation occurs.

\( d \) Show that if \( q_\lambda \) is a period 2 point of \( f_\lambda \), and \( q_\lambda \neq 0 \), then it satisfies the equation

\[ q_\lambda (\lambda e^{-q_\lambda} + 1) = 2 \log_e \lambda \]

Assuming (without proof) that this equation cannot have more than three solutions, show that there is no more than one orbit of prime period 2 no matter how large \( \lambda \) is made.

3. \( i \) Let \( f: S \to S \) be a discrete time dynamical system. Say what is meant by a fixed point (of \( f \)), a period \( n \) point and a prime period \( n \) point.

\( ii \) Let \( f: \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = ax + b \) where \( a \) and \( b \) are constants, and \( a \neq 1 \).

\( a \) Find the fixed point, \( x_* \), of \( f \).

\( b \) Find a formula for \( f^n(x) \) and use it to show that if \( |a| < 1 \) then \( f^n(x) \to x_* \) as \( n \to \infty \) for all \( x \). Also show that if \( |a| > 1 \) and \( x \neq x_* \), then \( |f^n(x)| \to \infty \) as \( n \to \infty \).

\( iii \) Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous and let \( O = \{x_0, x_1, \ldots, x_{n-1}\} \) be a prime period \( n \) orbit of \( f \). Let \( a \) and \( b \) be points in \( O \) such that \( a < b \) and there is no \( x \in O \) such that \( a < x < b \). (\( a \) and \( b \) are adjacent when the orbit is plotted on the real line.)

\( a \) Show that the number of points in \( O \) which are greater than \( a \) is one more than the number which are greater than \( b \).

\( b \) Use this fact to show that there is an integer \( m \), with \( 0 < m < n \), such that \( f^m(a) > a \) and \( f^m(b) < b \).
(c) Hence show that between $a$ and $b$ there is a periodic point of period less than $n$. (You will need to use the Intermediate Value Theorem.)

4. i) Let $\Sigma_2$ be the set of symbol sequences $\mathbf{s} = s_0s_1s_2 \ldots$ where $s_i = 0$ or 1 for all $i$, and say $\sigma: \Sigma_2 \to \Sigma_2$ is the shift map. Let the distance $d(\mathbf{s}, \mathbf{t})$ between points in $\Sigma_2$ be given by

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Consider the subset $T \subset \Sigma_2$ containing sequences such that if $s_i = 0$ then $s_{i+1} = 1$ (i.e. every 0 is followed by a 1).

(a) Show that if $\mathbf{s} \in T$ then $\sigma(\mathbf{s}) \in T$.

(b) Show that $T$ contains period $n$ points of $\sigma$ for every positive integer $n$. Show that periodic points are dense in $T$ (i.e. given any $\mathbf{t} \in T$ there is a periodic point $\mathbf{s} \in T$ arbitrarily close to $\mathbf{t}$).

(c) Show that if $\mathbf{s} \in T$ and $\mathbf{t} \notin T$ then for some $n$, $d(\sigma^n(\mathbf{s}), \sigma^n(\mathbf{t})) \geq 1/2$.

ii) Let $F_\mu: \mathbb{R} \to \mathbb{R}$ be the function $F_\mu(x) = \mu x (1-x)$, where $\mu > 4$, and let $I = [0, 1]$.

(a) Sketch a graph of $F_\mu$ on $I$ and use it to show that there are two closed intervals, $I_0$ and $I_1$, such that $F_\mu I_0 = I$ and $F_\mu I_1 = I$. Find $I_0$ and $I_1$.

(b) Let $\Lambda$ be the set of points $x$ in $I$ such that $F_\mu^n(x) \in I$ for all $n$. Show $\Lambda \subset I_0 \cup I_1$.

(c) Define $h: \Lambda \to \Sigma_2$ by $h(x) = \mathbf{s}$ where $s_i = 0$ if $F_\mu^i(x) \in I_0$ and $s_i = 1$ if $F_\mu^i(x) \in I_1$, for $i = 0, 1, \ldots$. For sufficiently large $\mu$ there is a constant $A > 1$ (depending on $\mu$) such that $|F_\mu'(x)| > A$ for all $x \in I_0 \cup I_1$. Show that for $\mu$ this large, $h$ is injective.

(d) Show that $\sigma \circ h = h \circ F_\mu$.

END OF PAPER