Date: Thursday, 29th January 1998

Time: 2.00–4.00 p.m.

Answer THREE questions.
1. \(i\) Consider continuous time dynamical systems of the form

\[ \frac{dx}{dt} = F(x,t) \]

where \(x \in \mathbb{R}^n\) and \(t \in \mathbb{R}\). Describe the difference between autonomous and non-autonomous systems. Explain what is meant by a ‘periodically forced’ system.

\(ii\) Show how an angle variable may be used to convert a periodically forced system into an autonomous one. Explain what is meant by the ‘augmented state space’. For the case \(x \in \mathbb{R}\), describe the nature of the augmented state space.

\(iii\) Consider the system

\[ \frac{dx}{dt} = -x + \alpha \cos t \]

where \(x \in \mathbb{R}\), and \(\alpha \in \mathbb{R}\) is a constant. Show, by substituting in to the differential equation, that the solution satisfying \(x(0) = x_0\) is

\[ x(t) = \left( x_0 - \frac{\alpha}{2} \right) e^{-t} + \frac{\alpha}{2} (\cos t + \sin t) \]

Using the angle variable \(\theta = t \mod 2\pi\) express this system as an autonomous, augmented system.

Let \(\Sigma\) be the subset of the augmented state space defined by \(\Sigma = \{(x,\theta) : \theta = 0\}\), and let \(\phi_{2\pi}\) be the time \(2\pi\) map of the augmented system. Show that \(\phi_{2\pi}\Sigma \subset \Sigma\). If \((x,0) \in \Sigma\) write down an expression for \(\phi_{2\pi}(x,0)\), and use it to show that \((\alpha/2,0)\) is a fixed point of \(\phi_{2\pi}\Sigma \rightarrow \Sigma\)
2.  
   i) Let $I \subset \mathbb{R}$ be the closed interval $[a, b]$ and $F: I \rightarrow I$ be continuous. Show that if $F(x) > x$ for all $x \in (a, b)$ and $F(b) = b$ then, for all $x \in I$, $F^n(x) \rightarrow b$ as $n \rightarrow \infty$. Sketch a cobweb diagram to illustrate this situation.

   ii) (a) Define $F_\mu: \mathbb{R} \rightarrow \mathbb{R}$ by $F_\mu(x) = \mu \cos x$, where $\mu > 0$. Using the Intermediate Value Theorem—which you should state clearly—show that $F$ has a fixed point in the interval $[0, \pi/2]$. Sketch a graph of $F_\mu$ to show that there are no other fixed points in this interval. Let $p_\mu$ be the fixed point of $F_\mu$ in $[0, \pi/2]$. For $\mu = 1$ show that $p_\mu$ is attracting, stating any results you use.

   (b) As $\mu$ is increased from 0, $p_\mu$ undergoes a period doubling bifurcation. If $\mu_*$ is the value of $\mu$ at which the bifurcation occurs, and $p_{\mu_*}$ the corresponding fixed point, show that

   \[
   \tan(p_{\mu_*}) = \frac{1}{p_{\mu_*}} \quad \text{and} \quad \mu_* = \sqrt{p_{\mu_*}^2 + 1}
   \]

3.  
   i) Let $f: X \rightarrow X$ and $g: Y \rightarrow Y$ be discrete time dynamical systems. Explain what a conjugacy between $f$ and $g$ is. Show that if $h: X \rightarrow Y$ is a conjugacy between $f$ and $g$, and $x \in X$ is a periodic point of $f$ with prime period $n$, then $h(x)$ is a prime period $n$ point of $g$.

   ii) Let $\Sigma_2$ be the set of symbol sequences $s = s_1 s_2 s_3 \ldots$ where $s_i = 0$ or 1 for all $i$, but where sequences ending in an infinite sequence of 1’s are excluded. Define the shift map $\sigma: \Sigma_2 \rightarrow \Sigma_2$. Show that repeating sequences correspond to periodic points of $\sigma$, and that there are $2^n - 1$ periodic points of $\sigma$ having period $n$.

   iii) Consider the doubling map $f: [0, 1) \rightarrow [0, 1)$ where

   \[
   f(x) = \begin{cases} 
   2x & \text{if } 0 \leq x < 1/2 \\
   2x - 1 & \text{if } 1/2 \leq x < 1
   \end{cases}
   \]

   Let $h: \Sigma_2 \rightarrow [0, 1)$ be defined by

   \[
   h(s) = \sum_{i=1}^{\infty} \frac{s_i}{2^i}
   \]

   Show that $h$ is a conjugacy between $f$ and $\sigma$. (Assume without proof that $h$ is invertible.)

   Prove that there is only one trajectory of $f$ which lies inside the interval $[1/4, 3/4)$, and find the trajectory.

P.T.O.
4. Let \( f: [0, 1] \rightarrow [0, 1] \) be continuous. Define a new function \( F: [0, 1] \rightarrow [0, 1] \), depending on \( f \), by

\[
F(x) = \begin{cases} 
\frac{1}{3}f(3x) + \frac{2}{3} & \text{if } 0 \leq x \leq 1/3 \\
-(f(1) + 2)(x - \frac{2}{3}) & \text{if } 1/3 \leq x \leq 2/3 \\
x - \frac{2}{3} & \text{if } 2/3 \leq x \leq 1
\end{cases}
\]

i) Show that the definitions of \( F \) on the intervals \([0, 1/3]\) and \([1/3, 2/3]\) agree at \( x = 1/3 \). Similarly, show that the definitions on the intervals \([1/3, 2/3]\) and \([2/3, 1]\) agree at \( x = 2/3 \).

ii) For the case \( f(x) = 4x(1-x) \), sketch a graph of the corresponding \( F(x) \). Your graph should indicate the values of \( F(x) \) at \( x = 0, 1/3, 2/3 \) and 1.

iii) For a general (continuous) \( f:[0, 1] \rightarrow [0, 1] \), show that if \( x \in [0, 1] \) and \( y = f(x) \), then \( F^2(x/3) = y/3 \).

iv) Let \( x \in [0, 1] \) be a periodic point of \( f \), with prime period \( n \).

(a) Show that \( x/3 \) is a periodic point of \( F \) with period \( 2n \).

(b) Show that it has prime period \( 2n \). (Hint: consider the possibilities that the prime period of \( x/3 \) is even and odd separately.)