2M2 Second problem sheet on Vector Calculus

1. Verify the identity \( \nabla \cdot (\phi \mathbf{V}) = \phi \nabla \cdot \mathbf{V} + \nabla \phi \cdot \mathbf{V} \) for \( \phi = x^2 - 3y + z \), \( \mathbf{V} = x^2z^3\mathbf{i} - (x^2 - y^2)\mathbf{j} \).

2. Calculate the line integral of the vector field \( \mathbf{F} = y^2\mathbf{i} + 2x(y + 1)\mathbf{j} \) from point \( A(1,1) \) to point \( B(2,2) \) along (a) the straight line \( AB \); (b) the path consisting of the horizontal segment \( AC \) and the vertical segment \( CB \) with point \( C \) having coordinates \( (2,1) \). What is the line integral around the closed loop (consisting of straight segments) \( ACBA \).

3. Calculate the line integral of the vector field \( \mathbf{F} = (3x^2 - yz)\mathbf{i} + z\mathbf{j} + xy\mathbf{k} \) from point \( (0,0,0) \) to point \( (1,1,1) \) along the path \( x = t, y = t^2, z = t^3 \).

4. Evaluate the line integral

\[
\int_{A}^{B} [x^2dx + (y^2 + x)dy]
\]

from point \( A(0,1) \) to point \( B(1,2) \) along (a) a straight line from \( A \) to \( B \) and (b) along the parabola defined by \( x = t, y = t^2 + 1 \).

5. Check the divergence theorem using the function \( \mathbf{F} = y^2\mathbf{i} + (2xy + z^2)\mathbf{j} + 2yz\mathbf{k} \) and the unit cube delimited by the planes \( x = 0, y = 0, z = 0, x = 1, y = 1 \) and \( z = 1 \).

6. Check Stokes’ theorem using the function \( \mathbf{F} = (2xz + 3y^2)\mathbf{j} + 4yz^2\mathbf{k} \) and the square surface whose corners are located at the origin and at the points \( (0,1,0), (0,1,1) \) and \( (0,0,1) \).