1. In each question write in matrix form $\dot{x} = Ax$ and form matrix $P$ with the eigenvectors as columns then the solution is

$$x(t) = Pe^{Dt}P^{-1}x(0)$$

where I have written $e^{Dt}$ as a shorthand for the diagonal matrix with $e^{\lambda_i t}$ on the diagonal. The eigenvalues $\lambda_i$ have to be in the same order as their eigenvectors in $P$. You should give the answer as $x_1, x_2$ in terms of $t$ as that is how the question was framed.

(a) Eigenvalues 6,1 $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, solution $x_1(t) = e^{2t} + e^{8t}$, $x_2(t) = -e^{2t} + e^{8t}$. In this case as the matrix is symmetric we could if we wanted have normalized the eigenvectors to have length one, and in that case we would have $P^T = P^{-1}$. But we we not asked for that in this question.

(b) Eigenvalues 8,2 $P = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}$, solution $x_1(t) = e^t + 4e^{6t}$, $x_2(t) = -4e^t + 4e^{6t}$

(c) Eigenvalues 1, $(3 \pm \sqrt{17})/3$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} + \frac{1}{8} (3 + \sqrt{17}) & \frac{1}{8} (3 - \sqrt{17}) \\ 0 & \frac{1}{8} (3 + \sqrt{17}) & 1 \end{pmatrix}$$

which is horrible except in this case you notice that the initial condition $x(0)$ is an eigenvector (at least in the corrected version of the sheet!) so the solution is easy. $x_1 = e^t, x_2(t) = 0, x_3(t) = 0$

2. (a) Let $\bar{y} = ax + bx^2$ be the approximate solution then residual

$$R(x) = \frac{d\bar{y}}{dx} + 2\bar{y} - 2$$

which in this case is $R(x) = a(1 - 2x) + 2b(x + x^2) - 1$

(b) We solve the equations $R(1/2) = 0, R(1) = 0$ which are

$$2a + \frac{3}{4}b = 2$$

$$3a + 4b = 2$$

Which gives $a = 10/7, b = -4/7$ and $\bar{y} = \frac{10}{7}x - \frac{4}{7}x^2$.

(c) (Galerkin= Weighted residual). In this case we solve

$$\int_0^1 xR(x) \, dx = 0, \int_0^1 x^2 R(x) \, dx = 0$$

which results in the equations

$$\frac{7}{5}a + \frac{7}{5}b = 1$$

$$\frac{8}{5}a + \frac{8}{5}b = \frac{2}{3}$$

with solution $a = 11/7, b = -5/7$ and $\bar{y} = \frac{11}{7}x - \frac{5}{7}x^2$. See the Mathematica notebook for a graph of the true and approximate solutions.