1. (a) \(-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0\), \(\lambda = 2, 2, 1\) eigenvectors for \(\lambda = 2\) are null vectors of

\[
A - 2I = \begin{pmatrix}
-2 & 0 & -2 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
\end{pmatrix}
\]

are \(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\) and \(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\) but any linear combination of those is also an eigenvector. For \(\lambda = 1\) the eigenvector is (any multiple of)

\[
\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}
\]

(b) \(-\lambda^3 + \frac{15\lambda^2}{8} - \frac{17\lambda}{16} + \frac{3}{16} = 0\) or \(-16\lambda^3 + 30\lambda^2 - 17\lambda + 3 = (\lambda - 1)(2\lambda - 1)(8\lambda - 3) = 0\), so \(\lambda = 1, 1/2, 3/8\).

\[
A - I = \begin{pmatrix}
-\frac{1}{2} & 0 & 0 \\
-1 & -\frac{5}{8} & 0 \\
2 & -4 & 0 \\
\end{pmatrix}
\]

which has null vector \(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\) which is the eigenvector for \(\lambda = 1\).

\[
A - \frac{1}{2}I = \begin{pmatrix}
0 & 0 & 0 \\
-1 & -\frac{5}{8} & 0 \\
2 & -4 & \frac{1}{2} \\
\end{pmatrix}
\]

It is maybe not so easy to see the null vector here even though it is clearly has rank one as a whole row is zero. using the trick I taught (determinants of minors) you you can see that an eigenvector for \(\lambda = 1/2\) is

\[
\begin{pmatrix}
-1/16 \\
1/2 \\
17/4 \\
\end{pmatrix}
\]

\[
A - \frac{3}{8}I = \begin{pmatrix}
\frac{1}{8} & 0 & 0 \\
-1 & 0 & 0 \\
2 & -4 & \frac{5}{8} \\
\end{pmatrix}
\]

This is easier as the two last columns are clearly multiples of one another. We get the eigenvector

\[
\begin{pmatrix}
0 \\
1 \\
32/5 \\
\end{pmatrix}
\]

(c) Eigenvalues are easy \(-(\lambda - 1)(\lambda - 2)^2 = 0\) so \(\lambda = 1, 2, 2\). The eigenvector for \(\lambda = 1\) is clearly \(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\) but notice that

\[
A - 2I = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\]
has rank one despite the repeated eigenvalue and hence here is only
one independent eigenvector for \( \lambda = 2 \) and it is
\[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

(d) You can probably see straight away that 2 is an eigenvalue. Characteristic equation is \(-\lambda^3 + 12\lambda^2 - 36\lambda + 3 = -(\lambda - 8)(\lambda - 2)^2 = 0\) so eigenvalues are \( \lambda = 8, 2, 2 \). This is a symmetric matrix so despite the repeated eigenvalue we expect to get three independent eigenvectors (and they should be orthogonal). Eigenvectors (in the order I gave the eigenvalues) are
\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
-1 \\
1 \\
0
\end{pmatrix}
\]

2. Easy to take dot product and we see the first eigenvector is orthogonal to
the other two. The problem is that our choice of the two vectors in the
null of \( A - 2I \) were not orthogonal. Lets look at that again

\[
A - 2I = \begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2
\end{pmatrix}
\]

we have to be a bit more imaginative. An easy way is to take the first
two eigenvectors we have, then we know the third must be orthogonal to
those so we could take the vector product
\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} \times \begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
1 \\
-2 \\
1
\end{pmatrix}
\]

and we see that this is clearly also an eigenvector for \( \lambda = 2 \). To make
an orthogonal matrix we need the columns not just to be orthogonal but
orthonormal, that is length one. So we divide by length and

\[
P = \begin{pmatrix}
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & 0 & -\frac{2}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\]

and it is easy to check \( P^TP = I \) and \( P^TAP = D \) where the eigenvalues
are on the diagonal of \( D \).

3. The eigenvalues are 3,2 an 1 and the eigenvectors are the columns of

\[
P = \begin{pmatrix}
1 & 2 & 1 \\
3 & 3 & 1 \\
4 & 3 & 1
\end{pmatrix}
\]

It is easy to check \( AP = PD \) with 3,2,1 on the diagonal of \( D \).
4. Here eigenvalues are 4,3,2 and

\[ P = \begin{pmatrix}
-1 & 0 & 0 \\
-3 & 1 & -2 \\
1 & 0 & 1
\end{pmatrix}. \]

It’s a little bit tedious to find \( P^{-1} \) by hand but it is

\[ P^{-1} = \begin{pmatrix}
-1 & 0 & 0 \\
-1 & 1 & 2 \\
1 & 0 & 1
\end{pmatrix} \]

5. So we have eigenvalues 8 and 2, with the (orthogonal) eigenvectors \( \left( \frac{1}{1} \right), \left( \frac{1}{-1} \right) \) which we make orthonormal by dividing by \( \sqrt{2} \) if we want. Now

\[ A = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
8 & 0 \\
0 & 2
\end{pmatrix} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}^{-1} \]

So

\[ A^6 = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
8 & 0 \\
0 & 2
\end{pmatrix}^6 \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}^{-1} \]

\[ = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
262144 & 0 \\
0 & 64
\end{pmatrix} \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{pmatrix} = \begin{pmatrix}
131104 & 131040 \\
131104 & 131104
\end{pmatrix} \]