Movement along a path in polar coordinates

Polar coordinates are particularly useful when there is a force between a particle and a central point. In polar coordinates in two dimensions we describe the position of a point by \((r, \theta)\), distance to the origin and angle anticlockwise from the initial direction, \(0 < r \) and either \(-\pi < \theta \leq \pi\) or \(0 \leq \theta < 2\pi\).

We have radial and angular unit vectors \(\hat{r}\) and \(\hat{\theta}\) which in Cartesian coordinates are \((\cos \theta, \sin \theta)\) and \((-\sin \theta, \cos \theta)\). Note the difference between this and the tangential-normal frame. They coincide for a particle going in a circle.

**NB** Although we can differentiate a path in Cartesian coordinates \((x(t), y(t))\) component-wise to give velocity, \((\dot{x}(t), \dot{y}(t))\) this doesn’t work for polar coordinates (look at the dimensions).

We can differentiate the Cartesian expressions for the polar unit vectors, remembering that \(r\) and \(\theta\) are functions of time (so we need to use chain and product rules). We get

\[
\frac{d}{dt} \hat{r} = \frac{d\theta}{dt} \hat{\theta}
\]

\[
\frac{d}{dt} \hat{\theta} = -\frac{1}{r} \frac{dr}{dt} \hat{r}
\]

Or in summary \(\dot{\hat{r}} = \hat{\theta}\) and \(\dot{\hat{\theta}} = -\hat{\theta}\). Now the position vector is \(\mathbf{r} = r\hat{r}\) which we differentiate with respect to time to give

\[
\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}
\]

Differentiating again gives acceleration

\[
\mathbf{a} = \ddot{\mathbf{v}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} - r\ddot{\theta}\hat{\theta}
\]

(use the chain and product rules here).

Ultimately

\[
\mathbf{a} = \ddot{\mathbf{v}} = \ddot{r}\hat{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\dddot{\theta} + 2\dot{r}\ddot{\theta})\hat{\theta}
\]

**Example 1** For circular motion \(\dot{r} = \ddot{r} = 0\) as the radius does not change. Hence

\[
\mathbf{a} = r\dddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}
\]

Using \(v = r\dot{\theta}\) and \(\dot{v} = r\ddot{\theta}\) this becomes

\[
\mathbf{a} = \frac{dv}{dt} \hat{\theta} - v \frac{d\theta}{dt} \hat{r}
\]
(Check this against the tangential-normal version.

**Example 2** Kepler’s second law.

**Note** If \( A \) is the area swept by the position vector then

\[
dA/dt = \frac{1}{2} r^2 \dot{\theta}
\]

**Hint**

\[
\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta})
\]

If \( \mathbf{a} = a \mathbf{\hat{r}} \) (that is the acceleration is *central*) show that the rate of change of area swept by the position vector is a constant.

We have

\[
\mathbf{a} = a \mathbf{\hat{r}} = (-r \dot{\theta}^2) \mathbf{\hat{r}} + (r \ddot{\theta} + 2r \dot{\theta}) \mathbf{\hat{\theta}}
\]

so \( \ddot{r} - r \dot{\theta}^2 = a \) and \( r \ddot{\theta} + 2r \dot{\theta} = 0 \) by equating components.

\( dA/dt \) is constant so \( d^2A/dt^2 = 0 \), now using the hint \( d(r^2 \dot{\theta})/dt = 0 \) is equivalent to

\[
0 = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = r \ddot{\theta} + 2r \dot{\theta}
\]

We now see that Kepler’s 2nd Law, constant rate of sweeping area, is equivalent to the acceleration (and hence force) being only in the radial direction. This is independent of the way the force depends on position. Next we’ll see that the force following an inverse square law leads to elliptical orbits.

First we need to know a bit about conic sections. The formula

\[
r = \frac{L}{1 + e \cos \theta}
\]

for \( L > 0 \) represents

- A circle if \( e = 0 \)
- An ellipse if \( |e| < 1 \)
- A parabola if \( |e| = 1 \)
- A hyperbola if \( |e| > 1 \)

That is a conic section symmetrical about the \( x \) axis (\( \theta = 0 \)). We can rotate the curve by replacing \( \theta \) by \( \theta + \alpha \). The constant \( e \) is called the eccentricity.

**Example 3** Newton’s inverse square law. Assuming that the path of a planet is a conic section (as above) with the Sun at the origin, and the force is central \( \mathbf{\hat{r}} = a \mathbf{\hat{r}} \) find a formula for \( a \) in terms of \( r \) and \( \theta \).