Alternative formulas for acceleration in one dimension.

\[
\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt}
\]

so

\[
a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx}
\]

This can be useful **Example 1** Acceleration as a function of time

If \(a = t - t^2\), and \(v = 0\) and \(x = 0\) when \(t = 0\) find \(x(t)\) and \(v(t)\)

Integrating we have

\[
v = t^2/2 - t^3/3
\]

where the constant of integration is found to be zero. Integrate again:

\[
x = t^3/6 - t^4/12
\]

(constant is zero again).

Sketching:

**Example 2** Acceleration as a function of position: \(a = 1 - x\) and again \(x = 0\) and \(v = 0\) when \(t = 0\), find \(v^2\) as a function of \(x\). Using \(a = v(dv/dx)\): \(v(dv/dx) = 1 - x\) sp \(\int v \, dv = \int (1 - x) \, dx\) so \(v^2/2 = x - x^2/2 + c\)

When \(t = 0\) \(x = 0\) and \(v = 0\) so \(1/2 = 1 - 1/2 + c\) and \(c = 0\) giving

\[
v^2 = x(2 - x)
\]

**Example 3** Acceleration as a function of velocity If acceleration is \(a = 1 - v^2\) and both \(x\) and \(v\) are zero when \(t = 0\) show that \(-\frac{1}{2} \ln(1 - v^2) = x\) and hence find \(v^2\) as a function of \(x\).

Using \(a = v \frac{dv}{dx}\):

\[
a = v \frac{dv}{dx} = 1 - v^2
\]

\[
\int \frac{v}{1 - v^2} \, dv = \int dx
\]

\[-\frac{1}{2} \ln(1 - v^2) = x + c\]
When $t = 0 \Rightarrow x = 0$ and $v = 0$ so that $-\frac{1}{2} \ln 1 = 0 + c$ hence $c = 0$ giving

$$\ln(1 - v^2) = -2x$$

Taking the exponential of both sides gives

$$v^2 = 1 - e^{-2x}$$

(Why is it most suitable to use $a = v \frac{dv}{dt}$ in this case?)

**Remember** we considered $a = a(v), v = v(x)$ in this example.

**Example 4** Acceleration as a function of velocity (again).

As above but now find $v$ as a function of $t$.

$$\frac{dv}{dt} = 1 - v^2$$

Integrating

$$\frac{1}{2} \ln \frac{1 + v}{1 - v} = t + c$$

(Check this). When $t = 0, v = 0$ so that $c = 0$ giving

$$\frac{1 + v}{1 - v} = 2t$$

Take the exponential and rearrange (check this)

$$v = \frac{e^{2t} - 1}{e^{2t} + 1} = \tanh t$$

sketching:

(-as an exercise, show that $x = \ln(\cosh t)$).