MATH31011/MATH41011/MATH61011:
Fourier Analysis and Lebesgue Integration

SOLUTIONS TO REVISION EXAMPLES

1. Uncountable: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z} \times [0,1]$
Countably infinite: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z} \times \mathbb{Q}$, $\bigcup_{n=1}^{\infty} \mathbb{N}^n$,
Finite: $\{r \in \mathbb{Q} \cap [0,1] : r = \frac{p}{q} \text{ for some } p, q \in \mathbb{N} \text{ and } 1 \leq q \leq 100\}$.

2. (a) Since all x_n are less or equal a, $\limsup_{n\to\infty} x_n < +\infty$ ($\limsup_{n\to\infty} x_n = +\infty$ would imply infinitely many $x_n > a$, and there is none). Assume $\limsup_{n\to\infty} x_n = c > a$. So $c \in \mathbb{R}$ and $\epsilon = \frac{c-a}{2} > 0$. By definition of lim sup, there there are (infinitely many) $n \in \mathbb{N}$ with $x_n > c - \epsilon > a$, contradiction. The lim inf part is similar.

(b) First note that by (a), since the two sequences are bounded (and since lim inf is always smaller or equal to lim sup), the lim inf and lim sup involved are in $\mathbb{R}$.

For a fixed $n \in \mathbb{N}$ and all $k \geq n$ we have $x_k + y_k \leq \sup_{k\geq n} x_k + \sup_{k\geq n} y_k$ so

$$\sup_{k\geq n}(x_k + y_k) \leq \sup_{k\geq n}(\sup_{k\geq n} x_k + \sup_{k\geq n} y_k) = \sup_{k\geq n} x_k + \sup_{k\geq n} y_k\,,$$

and taking limits as $n \to \infty$ and using Lemma 2.13 we obtain the result.

Alternatively, assume to the contrary that

$$\limsup_{n\to\infty}(x_n + y_n) > \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

and write

$$A = \limsup_{n\to\infty}(x_n + y_n), \quad B = \limsup_{n\to\infty} x_n, \quad C = \limsup_{n\to\infty} y_n\,.$$

Then $\epsilon = A - (B + C) > 0$ so from the definition of lim sup there are some $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $x_n < B + \frac{\epsilon}{4}$ and for all $n \geq N_2$ we have $y_n < C + \frac{\epsilon}{4}$ but also that there is $n \geq \max\{N_1, N_2\}$ such that $x_n + y_n > A - \frac{\epsilon}{4}$, so for this particular n

$$x_n + y_n > A - \frac{\epsilon}{4} > B + C + \frac{\epsilon}{2} > x_n + y_n\,,$$

contradiction.

The case with lim inf is similar.

Define $\quad x_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad$ and $\quad y_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

This gives the required example since $\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n = 1$, $\liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n = 0$ but $\limsup_{n\to\infty}(x_n + y_n) = \liminf_{n\to\infty}(x_n + y_n) = 1$.

3. Let $x \in \mathbb{R}$. Then

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n} \right) .$$

Open intervals are in $\mathcal{B}$ by definition and $\mathcal{B}$ is closed under countable intersections (Lemma 3.3). Hence $\{x\} \in \mathcal{B}$.

Alternatively, use $(-\infty, x)^c = [x, +\infty)$, $(x, +\infty)^c = (-\infty, x]$ and $\{x\} = [x, +\infty) \cap (-\infty, x]$ along with the fact that $\mathcal{B}$ is closed under complements and intersections.

The σ-algebra $\mathcal{C}$ generated by the collection of all sets consisting of a single real number contains all countable sets since it is closed under countable unions and countable sets can be obtained as countable unions of sets containing a single number each. $\mathcal{C}$ is also closed under complements and hence it also contains all *co-countable* sets (complements of countable sets). It has been shown on Example Sheet 3 (Example 3) that the collection of countable and co-countable subsets of $\mathbb{R}$ is a σ-algebra. Hence it must be the required σ-algebra generated by the collection $\{\{x\} : x \in \mathbb{R}\}$.

4. Let $\epsilon > 0$ and let $I_n = \left(\frac{1}{n} - \frac{\epsilon}{2^{n+2}}, \frac{1}{n} - \frac{\epsilon}{2^{n+2}}\right)$. We have

$$\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \bigcup_{n=1}^{\infty} I_n, \qquad \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2} < \epsilon$$

so the set is null.

5. *Sufficient conditions for measurability*: simple, continuous, equal to a measurable function μ-almost everywhere, the characteristic function of a Borel set, increasing.

[See Lemma 3,11, Lemma 3.12(ii), note that the characteristic function of a Borel set is simple and see Q6 on Examples 4 respectively.]

Not sufficient: takes finitely many values, bounded.

[For example, if $E \subset [0, 1]$ is a non-measurable set - note that by Theorem 3.7, proved in the Extra Reading section, such a set exists - then χ_E is not a measurable function.]

6. Let $a \in \mathbb{R}$. We have

$$\{x \in [0,1] : g(x) \le a\} =$$

$$= \begin{cases} \emptyset & \text{if } a < 0 \\ \{x \in [0,1] : f(x) \le 0\} & \text{if } a = 0 \text{ and} f(0) \le 0 \\ \{x \in [0,1] : f(x) \le 0\} \cup \{0\} & \text{if } a = 0 \text{ and} f(0) > 0 \\ \{x \in [0,1] : f(x) \le 0\} \cup [0,a] & \text{if } a > 0 \end{cases}$$

Note that the set in the last case is $\{x \in [0,1] : f(x) \le 0\} \cup \{x \in [0,a] : f(x) > 0\}$. Since f is measurable and $\mathcal{M}$ is a σ-algebra that contains $\{0\}$ and $[0,a]$, the set $\{x \in [0,1] : g(x) \le a\}$ is measurable. The number a was arbitrary, hence g measurable.

7. For example, $f(x) = 1/2$ for $x \in [0,1]$ and $g(x) = \chi_{[0,1] \setminus \mathbb{Q}}$.

8. Let $A_n = \{x \in [-\pi, \pi] : f_n(x) > a\}$, and let

$$B = \{x \in [-\pi, \pi] : f_n(x) \text{ does not converge to } f(x)\}.$$

The set $C = B \cup \bigcup_{n=1}^{\infty} A_n$ is null since it is a union of countably many null sets and hence null (see[1] Example 3 on Example Sheet 3). For $x \notin C$ we have $f_n(x) \le a$ for all n and $f_n(x) \to f(x)$, so $f(x) \le a$ for $x \notin C$, hence the result.

9. The functions f_n are continuous and hence measurable. For any $x \in [0,1]$ we have $e^{nx} \ge 1$ and $0 \le 1 + \sin(x) \le 2$ so $|f_n| \le 2$ for all n, and the constant function $g : [0,1] \to \mathbb{R}$ defined by $g(x) = 2$ is integrable. Moreover, for $x \in [0,1]$, $0 \le f_n(x) \le \frac{2}{(e^x)^n}$ so since for any $a > 1$ we have $\lim_{n\to\infty} \frac{1}{a^n} = 0$ and since $e^x > 1$ whenever $x > 0$, it follows that for $x \in (0,1]$ we have $\lim_{n\to\infty} f_n(x) = 0$, so $f_n \to 0$ as $n \to \infty$ μ-a.e. By the Dominated Convergence Theorem, $\lim_{n\to\infty} \int f_n \, d\mu = 0$.

10. The functions F_n are nonnegative, hence for each $x \in [-\pi, \pi]$, the sequence $\sum_{n=1}^{N} F_n(x)$ is increasing in N and as such it has a limit in $\mathbb{R}^*$ (possibly $+\infty$). Hence F is well defined. Let $\sum_{n=1}^{\infty} \|F_n\|_2 = A$. From Minkowski Inequality, for each $N \in \mathbb{N}$

$$\left\| \sum_{n=1}^{N} F_n \right\|_2 \le \sum_{n=1}^{N} \|F_n\|_2 \le A$$

and hence for each N

$$\int_{-\pi}^{\pi} \left(\sum_{n=1}^{N} F_n \right)^2 d\mu \;\le\; \pi A^2 .$$

[1] This is in part (b), case (i) of the proof. We remark that although countable unions of null sets are null, the collection $\mathcal{N}$ of null sets is *not* a σ-algebra since it is not closed under complements.

The functions $\left(\sum_{n=1}^{N} F_n\right)^2$ are nonnegative and form an increasing sequence (which converges to F^2) so by the Monotone Convergence Theorem,

$$\int_{-\pi}^{\pi} F^2 \, d\mu = \int_{-\pi}^{\pi} \lim_{n\to\infty} \left(\sum_{n=1}^{N} F_n\right)^2 = \lim_{N\to\infty} \int_{-\pi}^{\pi} \left(\sum_{n=1}^{N} F_n\right)^2 \, d\mu \ \leq \pi A^2 < \infty$$

which shows that $F \in L^2([-\pi,\pi],\mu,\mathbb{R})$ as required.

(b) F^2 is integrable and as such finite μ-a.e. (Lemma 3.19). Hence F is finite μ-a.e. The series $\sum_{n=1}^{\infty}(-1)^n F_n(x)$ converges absolutely (and hence converges) wherever $F(x) = \sum_{n=1}^{\infty} F_n(x)$ is finite, so H is defined by this series μ-a.e. Hence H_N converge to H as $N \to \infty$ μ-a.e. Also (both for x where $F(x) = \sum_{n=1}^{\infty} F_n(x)$ is finite and where it is not) we have

$$H_N^2(x) \leq \left|\sum_{n=1}^{N} F_n(x)\right|^2 \leq F^2(x), \quad H^2(x) \leq F^2(x)$$

so

$$(H - H_N)^2 = (H^2 - 2H_N H + H_N^2) \leq 4F^2$$

so since F^2 is integrable and $(H - H_N)^2$ converge to 0 μ-a.e, the Dominated Convergence Theorem yields the result.