1. (a) Classify the following sets as finite, countably infinite, or uncountable:

\[ N, \ Z, \ Q, \ R, \ \mathbb{C} \text{ (complex numbers)}, \ Z \times Q, \ \bigcup_{n=1}^{\infty} N^n, \ Z \times [0,1] \]
\[ \{ r \in Q \cap [0,1] : r = \frac{p}{q} \text{ for some } p, q \in \mathbb{N} \text{ and } 1 \leq q \leq 100 \} \]

2. (a) Let \( a, b \in \mathbb{R} \) and let \( x_n, n \geq 1 \) be a sequence of real numbers such that \( b \leq x_n \leq a \) for each \( n \). Prove that \( \limsup_{n \to \infty} x_n \leq a \) and \( b \leq \liminf_{n \to \infty} x_n \).

(b) Let \( x_n, y_n, n \geq 1 \) be bounded sequences of real numbers. Show that

\[ \limsup_{n \to \infty} (x_n + y_n) \leq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n, \]
\[ \liminf_{n \to \infty} x_n + \liminf_{n \to \infty} y_n \leq \liminf_{n \to \infty} (x_n + y_n). \]

Give examples of sequences \( x_n, y_n, n \geq 1 \) for which the above inequalities are strict.

3. Show that the Borel \( \sigma \)-algebra \( \mathcal{B} \) contains all sets \( \{x\} \) where \( x \in \mathbb{R} \). What is the \( \sigma \)-algebra \( \mathcal{C} \) that is generated by the collection of all \( \{x\}, x \in \mathbb{R} \)?

4. Use the definition of a null set to prove that the set \( \{ \frac{1}{n} : n \in \mathbb{N} \} \) is null.

5. Write down which of the following are sufficient conditions for a function \( f : [0,1] \to \mathbb{R}^* \) to be measurable:

   (a) \( f \) is simple,
   (b) \( f \) is continuous,
   (c) \( f \) takes finitely many values
   (d) \( f \) is equal to a measurable function \( \mu \)-almost everywhere,
   (e) \( f \) is the characteristic function of a Borel set.
   (f) \( f \) is increasing,
   (g) \( f \) is bounded.

6. Use the definition of a measurable set to prove that if \( f : [0,1] \to \mathbb{R} \) is measurable then so is the function \( g : [0,1] \to \mathbb{R} \) defined by

\[ g(x) = \begin{cases} 
0 & \text{if } f(x) \leq 0, \\
x & \text{otherwise.} 
\end{cases} \]
7. Give an example of $f, g : [0, 1] \to \mathbb{R}^*$ such that $f \leq g$ $\mu$-a.e. but $f(x) > g(x)$ for infinitely many $x \in [0, 1]$.

8. Let $a \in \mathbb{R}$ and $f_n, f : [-\pi, \pi] \to \mathbb{R}$ be such that $f_n \to f$ $\mu$-a.e. and for each $n \in \mathbb{N}$, $f_n \leq a$ $\mu$-a.e. Show that $f \leq a$ $\mu$-a.e.

9. Use the dominated convergence theorem to find $\lim_{n \to \infty} \int f_n \, d\mu$, where the functions $f_n : [0, 1] \to \mathbb{R}$ are defined by

$$f_n(x) = \frac{1 + \sin(x)}{e^{nx}}$$

10. (a) Let $F_n : [-\pi, \pi] \to \mathbb{R}$, $n \geq 1$ be a sequence of non-negative functions from $L^2([-\pi, \pi], \mu, \mathbb{R})$ such that $\sum_{n=1}^{\infty} \|F_n\|_2 < +\infty$. Justify why $F : [-\pi, \pi] \to \mathbb{R}^*$,

$$F(x) = \sum_{n=1}^{\infty} F_n(x)$$

is a well defined function and use the Monotone Convergence Theorem to prove that $F \in L^2([-\pi, \pi], \mu, \mathbb{R})$.

(b) With $F_n, n \geq 1$ as in (a), show that $\sum_{n=1}^{\infty} (-1)^n F_n(x)$ converges (to a finite value) for $\mu$-a.e. $x$. Defining $H : [\pi, \pi] \to \mathbb{R}$ by

$$H(x) = \begin{cases} \sum_{n=1}^{\infty} (-1)^n F_n(x) & \text{if this series converges,} \\ 0 & \text{otherwise,} \end{cases}$$

and $H_N(x) = \sum_{n=1}^{N} (-1)^n F_n(x)$, show that $H \in L^2([-\pi, \pi], \mu, \mathbb{R})$ and

$$\|H - H_N\|_2 \to 0 \text{ as } N \to \infty.$$