MATH31011/MATH41011/MATH61011:
Fourier Analysis and Lebesgue Integration

REVISION EXAMPLE SHEET

1. (a) Classify the following sets as finite, countably infinite, or uncountable:

$$\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{R},\ \mathbb{C}\,(\text{complex numbers}),\ \mathbb{Z}\times\mathbb{Q},\ \bigcup_{n=1}^{\infty}\mathbb{N}^n,\ \mathbb{Z}\times[0,1]$$

$$\{r\in\mathbb{Q}\cap[0,1] : r=\frac{p}{q}\ \text{for some}\ p,q\in\mathbb{N}\,\text{and}\,1\le q\le 100\},$$

2. (a) Let $a,b\in\mathbb{R}$ and let x_n, $n\ge 1$ be a sequence of real numbers such that $b\le x_n\le a$ for each n. Prove that $\limsup_{n\to\infty} x_n\le a$ and $b\le\liminf_{n\to\infty} x_n$.

(b) Let $x_n, y_n,\ n\ge 1$ be bounded sequences of real numbers. Show that

$$\limsup_{n\to\infty}(x_n+y_n)\le\limsup_{n\to\infty} x_n+\limsup_{n\to\infty} y_n\,,$$
$$\liminf_{n\to\infty} x_n+\liminf_{n\to\infty} y_n\le\liminf_{n\to\infty}(x_n+y_n)\,.$$

Give examples of sequences $x_n, y_n,\ n\ge 1$ for which the above inequalities are strict.

3. Show that the Borel σ-algebra $\mathcal{B}$ contains all sets $\{x\}$ where $x\in\mathbb{R}$. What is the σ-algebra $\mathcal{C}$ that is generated by the collection of all $\{x\}$, $x\in\mathbb{R}$?

4. Use the definition of a null set to prove that the set $\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$ is null.

5. Write down which of the following are sufficient conditions for a function $f:[0,1]\to\mathbb{R}^*$ to be measurable:

(a) f is simple,
(b) f is continuous,
(c) f takes finitely many values
(d) f is equal to a measurable function μ-almost everywhere,
(e) f is the characteristic function of a Borel set.
(f) f is increasing,
(g) f is bounded.

6. Use the definition of a measurable set to prove that if $f:[0,1]\to\mathbb{R}$ is measurable then so is the function $g:[0,1]\to\mathbb{R}$ defined by

$$g(x)=\begin{cases}0 & \text{if } f(x)\le 0,\\ x & \text{otherwise.}\end{cases}$$

7. Give an example of $f, g : [0,1] \to \mathbb{R}^*$ such that $f \leq g$ μ-a.e. but $f(x) > g(x)$ for infinitely many $x \in [0,1]$.

8. Let $a \in \mathbb{R}$ and $f_n, f : [-\pi, \pi] \to \mathbb{R}$ be such that $f_n \to f$ μ-a.e. and for each $n \in \mathbb{N}$, $f_n \leq a$ μ-a.e. Show that $f \leq a$ μ-a.e.

9. Use the dominated convergence theorem to find $\lim_{n\to\infty} \int f_n \, d\mu$, where the functions $f_n : [0,1] \to \mathbb{R}$ are defined by

$$f_n(x) = \frac{1 + \sin(x)}{e^{nx}}$$

10. (a) Let $F_n : [-\pi, \pi] \to \mathbb{R},\ n \geq 1$ be a sequence of non-negative functions from $L^2([-\pi, \pi], \mu, \mathbb{R})$ such that $\sum_{n=1}^{\infty} \|F_n\|_2 < +\infty$. Justify why $F : [-\pi, \pi] \to \mathbb{R}^*$,

$$F(x) = \sum_{n=1}^{\infty} F_n(x)$$

is a well defined function and use the Monotone Convergence Theorem to prove that $F \in L^2([-\pi, \pi], \mu, \mathbb{R})$.

(b) With F_n, $n \geq 1$ as in (a), show that $\sum_{n=1}^{\infty} (-1)^n F_n(x)$ converges (to a finite value) for μ-a.e. x. Defining $H : [\pi, \pi] \to \mathbb{R}$ by

$$H(x) = \begin{cases} \sum_{n=1}^{\infty} (-1)^n F_n(x) & \text{if this series converges,} \\ 0 & \text{otherwise,} \end{cases}$$

and $H_N(x) = \sum_{n=1}^{N} (-1)^n F_n(x)$, show that $H \in L^2([-\pi, \pi], \mu, \mathbb{R})$ and

$$\|H - H_N\|_2 \to 0 \quad \text{as} \quad N \to \infty\,.$$