Recall:

**Theorem (Fundamental Theorem of Calculus for Riemann integral)**  
Let $f : [a, b] \to \mathbb{R}$ be continuous.

(a) Define $F : [a, b] \to \mathbb{R}$ by  
\[ F(x) = \int_{a}^{x} f(t)dt \]
Then $F$ is differentiable on $(a, b)$ and $F'(x) = f(x)$ for each $x \in (a, b)$.

(b) (Newton-Leibniz Axiom) Assume that $F : [a, b] \to \mathbb{R}$ is such that $F'(x) = f(x)$ for each $x \in [a, b]$. Then  
\[ \int_{a}^{x} f(t)dt = F(x) - F(a) \]
for each $x \in [a, b]$.

**Notes** In (a), if $f$ is not continuous but merely Riemann integrable then $F$ does not need to be differentiable at every $x \in (a, b)$. In (b) it suffices to assume Riemann integrability of $f$ on $[a, x]$ for each $x$ in place of its continuity.

We have a similar result for Lebesgue integral:

**Theorem** Let $f : [a, b] \to \mathbb{R}$ be integrable.

(a) Define $F : [a, b] \to \mathbb{R}$ by  
\[ F(x) = \int_{a}^{x} f d\mu = \int_{a}^{b} f \chi_{[a, x]} d\mu \]
Then $F$ is differentiable $\mu$-a.e. and $F' = f \mu$-a.e.
(b) Assume that $F : [a, b] \to \mathbb{R}$ is such that $F'(x) = f(x)$ for each $x \in [a, b]$. Then

$$\int_a^x f \, d\mu = F(x) - F(a)$$

for each $x \in [a, b]$.

**Proof:** omitted.

However, the name *Fundamental Theorem of Calculus for Lebesgue Integration* is usually given to the theorem below.

**Definition** $F : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for any $\epsilon > 0$ there is $\delta > 0$ such that whenever $[x_i, y_i], i = 1, \ldots, k$ are pairwise disjoint subintervals of $[a, b]$ such that

$$\sum_{i=1}^k (x_i - y_i) < \delta$$

then

$$\sum_{i=1}^k |F(x_i) - F(y_i)| < \epsilon.$$ 

Absolute continuity implies continuity and even uniform continuity.

Absolutely continuous functions are precisely the indefinite integrals of Lebesgue integrable functions in the following sense:

**Theorem** $F$ is absolutely continuous just when there is an integrable function $g$ such that for all $x \in [a, b]$

$$F(x) - F(a) = \int_a^x g \, d\mu .$$

(Moreover, if it is the case, $g = F' \mu$-a.e.)

**Proof:** omitted.