MATH31011/MATH41011/MATH61011: Fourier Analysis and Lebesgue Integration

Appendix II to Chapter 3: Fundamental Theorem of Calculus

This material is not examinable.

Recall:
Theorem (*Fundamental Theorem of Calculus for Riemann integral*)
Let $f : [a, b] \to \mathbb{R}$ be continuous.

(a) Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt$$

Then F is differentiable on (a, b) and $F'(x) = f(x)$ for each $x \in (a, b)$.

(b) (*Newton-Leibniz Axiom*) Assume that $F : [a, b] \to \mathbb{R}$ is such that $F'(x) = f(x)$ for each $x \in [a, b]$. Then

$$\int_a^x f(t)dt = F(x) - F(a)$$

for each $x \in [a, b]$.

Notes In (a), if f is not continuous but merely Riemann integrable then F does not need to be differentiable at every $x \in (a, b)$. In (b) it suffices to assume Riemann integrability of f on $[a, x]$ for each x in place of its continuity.

We have a similar result for Lebesgue Integral:

Theorem Let $f : [a, b] \to \mathbb{R}$ be integrable.

(a) Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f d\mu \ \left(= \int_a^b f \chi_{[a,x]} d\mu \right)$$

Then F is differentiable μ-a.e. and $F' = f$ μ-a.e.

(b) Assume that $F : [a, b] \to \mathbb{R}$ is such that $F'(x) = f(x)$ for each $x \in [a, b]$..
Then

$$\int_a^x f d\mu = F(x) - F(a)$$

for each $x \in [a, b]$.

Proof: omitted.

However, the name *Fundamental Theorem of Calculus for Lebesgue Integration* is usually given to the theorem below.

Definition $F : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for any $\epsilon > 0$ there is $\delta > 0$ such that whenever $[x_i, y_i]$, $i = 1, \dots, k$ are pairwise disjoint subintervals of $[a, b]$ such that

$$\sum_{i=1}^{k} |x_i - y_i| < \delta$$

then

$$\sum_{i=1}^{k} |F(x_i) - F(y_i)| < \epsilon .$$

Absolute continuity implies continuity and even uniform continuity,

Absolutely continuous functions are precisely the indefinite integrals of Lebesgue integrable functions in the following sense:

Theorem F is absolutely continuous just when there is an integrable function g such that for all $x \in [a, b]$

$$F(x) - F(a) = \int_a^x g \, d\mu .$$

(Moreover, if it is the case, $g = F'$ μ-a.e.)

Proof: omitted.