MATH31011/MATH41011/MATH61011: FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Chapter 4: Fourier Series and $L^2([-\pi,\pi],\mu)$

Square Integrable Functions

Definition. *Let $f : [-\pi, \pi] \to \mathbb{R}$ be measurable. We say that f is square integrable if $|f|^2$ is integrable, i.e., if*

$$\int_{-\pi}^{\pi} |f|^2 \, d\mu < +\infty.$$

For future use, we shall write

$$\|f\|_2 = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 \, d\mu \right)^{1/2}$$

and we shall call this the L^2-norm of f (pronounced "L-two", not "L-squared").

Remarks.
(i) Of course, as f is real valued, $|f|^2 = f^2$, so we could just write $\int_{-\pi}^{\pi} f^2 \, d\mu$. However, the definition can be extended to complex valued functions when we really do want $|f|^2$.
(ii) The factor $1/\pi$ is not essential but is introduced to be convenient for the calculations with Fourier series which occur later in the chapter.

We'd like to use $\|\cdot\|_2$ to give a distance (more formally, a metric) on the set of square integrable functions by

$$\mathrm{dist}(f,g) = \|f-g\|_2 = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |f-g|^2 \, d\mu \right)^{1/2}.$$

However, if $f = g$ μ-a.e. then $\int_{-\pi}^{\pi} |f-g|^2 \, d\mu = 0$, so we would have distinct functions lying distance zero apart. To avoid this, we consider equivalence classes of square integrable functions under the equivalence relation

$$f \sim g \quad \Longleftrightarrow \quad f = g \ \mu\text{-a.e.}.$$

Thus an equivalence class has the form

$$[f] = \{g : [-\pi,\pi] \to \mathbb{R} : f = g \ \mu\text{-a.e.}\},$$

for some square integrable function $f : [-\pi, \pi] \to \mathbb{R}$.

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Definition. *We write $L^2([-\pi,\pi],\mu,\mathbb{R})$ for the set of equivalence classes of square integrable functions on $[-\pi,\pi]$.*

After having made this definition, we shall proceed more informally and write $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ (rather than the pedantic $[f] \in L^2([-\pi,\pi],\mu,\mathbb{R})$) whenever $f : [-\pi,\pi] \to \mathbb{R}$ is square integrable. However, bear in mind that two functions which are equal almost everywhere are now considered to be the same. We may also write $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ when $f : [-\pi,\pi] \to \mathbb{R}^*$ is such that $\int_{-\pi}^{\pi} f^2\,d\mu < +\infty$ but note that such an f must be finite μ-a.e. and hence equal μ-a.e. to a function with values in $\mathbb{R}$.

Now we shall show that $L^2([-\pi,\pi],\mu,\mathbb{R})$ is a vector space and that $d(f,g) = \|f-g\|_2$ is a metric on $L^2([-\pi,\pi],\mu,\mathbb{R})$. First we need a technical result.

Lemma 4.1. *If $f,g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ then*

$$\frac{2}{\pi}\int_{-\pi}^{\pi} fg\; d\mu \leq \|f\|_2^2 + \|g\|_2^2,$$

with equality if and only if $f = g$ μ-a.e. (i.e. if f and g are the same element of $L^2([-\pi,\pi],\mu,\mathbb{R})$).

Proof. We have

$$0 \leq (f(x)-g(x))^2 = f(x)^2 - 2f(x)g(x) + g(x)^2,$$

so

$$2f(x)g(x) \leq f(x)^2 + g(x)^2 \qquad (*)$$

Note that fg is integrable since, as above but for $(|f|-|g|)^2$, $|fg| \leq \frac{f^2+g^2}{2}$. Multiplying (*) by $1/\pi$ and integrating gives the first statement. Clearly, we get equality if and only if

$$\int_{-\pi}^{\pi} (f-g)^2\; d\mu = 0.$$

By Proposition 3.21, this holds if and only if $(f-g)^2 = 0$ μ-a.e, i.e., $f = g$ μ-a.e. □

Corollary. *If $f,g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ then*

$$\frac{2}{\pi}\int_{-\pi}^{\pi} |fg|\; d\mu \leq \|f\|_2^2 + \|g\|_2^2,$$

with equality if and only if $|f| = |g|$ μ-a.e.

Lemma 4.2. *If $f,g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ then*

(i) **Hölder Inequality***:*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |fg|\; d\mu \leq \|f\|_2\|g\|_2,$$

with equality if and only if for some $c \in \mathbb{R}$, $|f| = c|g|$ μ-a.e. or $\|f\|_2\|g\|_2 = 0$.

(ii) **Cauchy-Schwarz Inequality***:*

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi} fg\; d\mu\right| \leq \|f\|_2\|g\|_2,$$

with equality if and only if for some $c \in \mathbb{R}$, $f = cg$ μ-a.e. or $\|f\|_2\|g\|_2 = 0$.

Proof. Exercise. □

Lemma 4.3. *If $f, g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ then so is $f+g$ and*

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2\,.$$

Proof. Clearly $f+g$ is measurable. Also

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}(f+g)^2\;d\mu &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f^2+2fg+g^2)\;d\mu \\ &\leq \frac{1}{\pi}\int_{-\pi}^{\pi}(f^2+2|fg|+g^2)\;d\mu \\ &\leq \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2 \\ &= (\|f\|_2+\|g\|_2)^2\end{aligned}$$

(where we have used Hölder's inequality for the second inequality). Hence $f+g \in L^2([-\pi,\pi],\mu,\mathbb{R})$ and taking square roots gives the inequality (called Minkowski's Inequality). $\square$

Corollary. *$L^2([-\pi,\pi],\mu,\mathbb{R})$ is a vector space over $\mathbb{R}$.*

Proof. It is trivial that if $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ and $c \in \mathbb{R}$ then $cf \in L^2([-\pi,\pi],\mu,\mathbb{R})$ (and $\|cf\|_2 = |c|\|f\|_2$). By the above lemma, if $f, g \in L^2([-\pi,\pi],\mu,\mathbb{R})$, so is $f+g \in L^2([-\pi,\pi],\mu,\mathbb{R})$. $\square$

Definition. *A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that, for $x, y, z \in X$,*

1. *$d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;*
2. *$d(x,y) = d(y,x)$;*
3. *$d(x,z) \leq d(x,y) + d(y,z)$.*

Theorem 4.4. *$d(f,g) = \|f-g\|_2$ is a metric on $L^2([-\pi,\pi],\mu,\mathbb{R})$.*

Proof. It is immediate from its definition that $\|f-g\|_2 \geq 0$. Suppose that

$$\|f-g\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}|f-g|^2\;d\mu = 0.$$

Applying Proposition 3.21, we see that $|f-g|^2 = 0$ μ-a.e., i.e., $f = g$ μ-a.e., so f and g represent the same element of $L^2([-\pi,\pi],\mu,\mathbb{R})$. This shows (1).

Condition (2) follows immediately from the definition.

For (3), assume that $f, g, h \in L^2([-\pi,\pi],\mu,\mathbb{R})$. By Minkowski's Inequality,

$$d(f,h) = \|f-h\|_2 = \|(f-g)+(g-h)\|_2 \leq \|f-g\|_2 + \|g-h\|_2 = d(f,g) + d(g,h)$$

as required. $\square$

The next important result says that square integrable functions may be approximated arbitrarily well by continuous functions with respect to $\|\cdot\|_2$.

Theorem 4.5. *Continuous functions are* $\|\cdot\|_2$*-dense in* $L^2([-\pi,\pi],\mu,\mathbb{R})$*. In other words, given* $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ *and* $\epsilon > 0$*, we can find a continuous function* $g : [-\pi,\pi] \to \mathbb{R}$ *such that* $\|f - g\|_2 < \epsilon$*.*

The proof is given in an appendix and is not examinable.

The following slight strengthening will be needed later on.

Corollary 4.6. *Given* $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ *and* $\epsilon > 0$*, we can find a continuous function* $g : [-\pi,\pi] \to \mathbb{R}$ *such that* $g(-\pi) = g(\pi)$ *and* $\|f - g\|_2 < \epsilon$*.*

Proof. Exercise. (Hint: use Theorem 4.5 to find a continuous function $h : [-\pi,\pi] \to \mathbb{R}$ such that $\|f - h\|_2 < \epsilon/2$. Modify h on $[-\pi,-\pi+\delta] \cup [\pi-\delta,\pi]$, for an appropriately small $\delta > 0$ to obtain a continuous function $g : [-\pi,\pi] \to \mathbb{R}$ with the required properties.) □

Definition. *We say that a sequence of functions* $f_n \in L^2([-\pi,\pi],\mu,\mathbb{R})$ *converges to* $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ *if*

$$\lim_{n\to+\infty} \|f_n - f\|_2 = 0.$$

Definition. *Recall that a metric space* (X,d) *is said to be complete if every Cauchy sequence converges to a point in* X*. (A sequence* $x_n \in X$ *is a Cauchy sequence if, for all* $\epsilon > 0$*, there exists* $N \geq 1$ *such that if* $n, m \geq N$ *then* $d(x_n, x_m) < \epsilon$*.*

Theorem 4.7. $L^2([-\pi,\pi],\mu,\mathbb{R})$ *is complete.*

Proof. Let f_n, $n \geq 1$, be a Cauchy sequence in $L^2([-\pi,\pi],\mu,\mathbb{R})$ with respect to the metric $d(f,g) = \|f - g\|_2$ determined by the norm $\|\cdot\|_2$. By definition, this means that, given $\epsilon > 0$, we can find an integer $N \geq 1$ such that

$$n, m \geq N \quad \Longrightarrow \quad \|f_n - f_m\|_2 < \epsilon.$$

Applying this definition with $\epsilon = 2^{-i}$, $i = 1, 2, \ldots$, we can find a increasing sequence of positive integers N_i such that

$$n, m \geq N_i \quad \Longrightarrow \quad \|f_n - f_m\|_2 < \frac{1}{2^i}.$$

Define functions $g_0 = 0$ and $g_i = f_{N_i}$, $i \geq 1$. Then

$$\|g_{i+1} - g_i\|_2 = \|f_{N_{i+1}} - f_{N_i}\|_2 < \frac{1}{2^i}$$

for $i \geq 1$. Thus, by the Comparison Test, the series

$$\sum_{i=0}^{\infty} \|g_{i+1} - g_i\|_2$$

converges. Let the sum be denoted by S.

Now consider a new sequence of functions h_n, $n \geq 1$, defined by

$$h_n(x) = \sum_{i=0}^{n-1} |g_{i+1}(x) - g_i(x)|.$$

For any fixed x, the sequence of numbers $h_n(x)$ is increasing, so we can defined

$$h(x) := \lim_{n\to+\infty} h_n(x) \in \mathbb{R} \cup \{+\infty\}.$$

Thus we have a measurable function $h : [-\pi, \pi] \to \mathbb{R}^*$.

We want to show that $h \in L^2([-\pi, \pi], \mu, \mathbb{R})$. First note that

$$\|h_n\|_2 \leq \sum_{i=0}^{n-1} \|g_{i+1} - g_i\|_2 \leq S,$$

so that

$$\int_{-\pi}^{\pi} h_n^2 \, d\mu = \pi \|h_n\|^2 \leq \pi S^2.$$

Since h_n^2 is an increasing sequence of non-negative measurable functions converging pointwise to h^2, the Monotone Convergence Theorem tells us that

$$\int_{-\pi}^{\pi} h^2 \, d\mu = \lim_{n\to+\infty} \int_{-\pi}^{\pi} h_n^2 \, d\mu \leq \pi S^2,$$

so $h^2 = |h|^2$ is integrable. Thus, $h \in L^2([-\pi, \pi], \mu, \mathbb{R})$, as we claimed.

Since h^2 is integrable it is finite μ-a.e. Thus, h is finite μ-a.e. For each $x \in [-\pi, \pi]$ for which $h(x)$ is finite, the series of real numbers

$$\sum_{i=0}^{\infty} (g_{i+1}(x) - g_i(x))$$

converges absolutely and hence converges. We will denote its sum by $g(x)$. For x with $h(x) = +\infty$, we set $g(x) = 0$.

Note that

$$\sum_{i=0}^{n-1} (g_{i+1}(x) - g_i(x)) = g_n(x) - g_0(x) = g_n(x).$$

Hence,

$$\lim_{n\to+\infty} g_n(x) = \lim_{n\to+\infty} \sum_{i=0}^{n-1} (g_{i+1}(x) - g_i(x)) = g(x),$$

for μ-a.e. x. Moreover,

$$\begin{aligned} |g(x)| &= \lim_{n\to+\infty} |g_n(x)| \\ &\leq \lim_{n\to+\infty} \sum_{i=0}^{n-1} |g_{i+1}(x) - g_i(x)| \\ &= \lim_{n\to+\infty} h_n(x) = h(x), \end{aligned}$$

for μ-a.e. x. Thus $|g(x)|^2 \leq |h(x)|^2$ for μ-a.e. x and so $|g|^2$ is integrable, giving $g \in L^2([-\pi, \pi], \mu, \mathbb{R})$.

We also observe that

$$|g(x) - g_n(x)|^2 \leq (|g(x)| + |g_n(x)|)^2 \leq (2h(x))^2.$$

Since $\lim_{n\to+\infty} |g(x) - g_n(x)|^2 = 0$ for μ-a.e. x, the Dominated Convergence Theorem tells us that

$$\lim_{n\to+\infty} \int_{-\pi}^{\pi} |g - g_n|^2 \, d\mu = 0.$$

This implies that

$$\lim_{n\to+\infty} \|g - g_n\|_2 = 0.$$

Hence, given $\epsilon > 0$, we can choose an i sufficiently large that $\|g - g_i\|_2 < \epsilon/2$ and $2^{-i} < \epsilon/2$. Recall that $g_i = f_{N_i}$. Thus, whenever $n \geq N_i$, we have

$$\|g - f_n\|_2 \leq \|g - g_i\|_2 + \|g_i - f_n\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{n\to+\infty} \|g - f_n\|_2 = 0$, i.e., the sequence f_n converges in the space $L^2([-\pi, \pi], \mu, \mathbb{R})$. □

Inner Products and Hilbert Spaces

Definition. Let V be a vector space over $\mathbb{R}$. A map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is called an *inner product* if, for all $u, v, w \in V$ and $a, b \in \mathbb{R}$,

(1) $\langle u, v \rangle = \langle v, u \rangle$;
(2) $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$;
(3) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$.

Lemma 4.8. *The formula*

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} fg \, d\mu$$

defines an inner product on the vector space $L^2([-\pi, \pi], \mu, \mathbb{R})$ *and*

$$\|f\|_2 = \langle f, f \rangle^{1/2}.$$

Proof. Parts (1) and (2) of the definition of inner product are easy to check. Part (3) is equivalent to the statement that $\|f\|_2 = 0$ if and only if f represents the 0 element in $L^2([-\pi, \pi], \mu, \mathbb{R})$, which follows from Proposition 3.21. □

Thus the metric on $L^2([-\pi, \pi], \mu, \mathbb{R})$ is obtained from this inner product by

$$d(f, g) = \|f - g\|_2 = \langle f - g, f - g \rangle^{1/2}.$$

(In fact, any inner product defines a metric in this way.)

Definition A vector space with an inner product which is complete with respect to the associated metric is called a *Hilbert space.*

Thus we have already proved:

Theorem 4.9. $L^2([-\pi, \pi], \mu, \mathbb{R})$ *is a Hilbert space.*

Definition. Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. We shall write $\| \cdot \|$ for the associated norm $\|v\| = \langle v, v \rangle^{1/2}$. We say that a collection of vectors $\{v_n\}$ in V is *orthogonal* if

$$\langle v_n, v_m \rangle = 0 \quad \text{whenever } n \neq m.$$

We say they are *orthonormal* if, in addition,

$$\|v_n\| = 1 \quad \text{for all } n.$$

We will use couple of standard results about orthogonal/orthonormal vectors.

Lemma 4.10. *Let $\{v_k\}_{k=1}^n$ be a finite orthogonal family in a vector space V and let $c_1, \ldots, c_n \in \mathbb{R}$. Then*

$$\left\| \sum_{k=1}^n c_k v_k \right\|^2 = \sum_{k=1}^n c_k^2 \|v_k\|^2.$$

Proof. Note that, by the definition of inner product and orthogonality,

$$\|c_1 v_1 + c_2 v_2\|^2 = \langle c_1 v_1 + c_2 v_2, c_1 v_1 + c_2 v_2 \rangle = c_1^2 \langle v_1, v_1 \rangle + 2c_1 c_2 \langle v_1, v_2 \rangle + c_2^2 \langle v_2, v_2 \rangle$$

$$= c_1^2 \|v_1\|^2 + c_2^2 \|v_2\|^2.$$

It is left as an exercise to complete the proof by induction. □

Lemma 4.11. *Let $\{v_k\}_{k=1}^n$ be a finite orthonormal family in a vector space V. Then, for $w \in V$, the minimum value of*

$$\left\| w - \sum_{k=1}^n c_k v_k \right\|$$

over all choices of $c_1, \ldots, c_n \in \mathbb{R}$ occurs when $c_k = \langle w, v_k \rangle$.

Proof. Let $c_1, \ldots, c_n$ be arbitrary real numbers and set $a_k = \langle w, v_k \rangle$. Write

$$u = \sum_{k=1}^n a_k v_k \quad \text{and} \quad v = \sum_{k=1}^n c_k v_k.$$

By the preceding lemma,

$$\|u\|^2 = \sum_{k=1}^n a_k^2 \quad \text{and} \quad \|v\|^2 = \sum_{k=1}^n c_k^2.$$

Also

$$\langle w, v \rangle = \left\langle w, \sum_{k=1}^n c_k v_k \right\rangle = \sum_{k=1}^n c_k \langle w, v_k \rangle = \sum_{k=1}^n c_k a_k.$$

Thus

$$\begin{aligned}\|w-v\|^2 &= \langle w-v, w-v\rangle \\ &= \|w\|^2 - 2\langle w,v\rangle + \|v\|^2 \\ &= \|w\|^2 - 2\sum_{k=1}^{n} c_k a_k + \sum_{k=1}^{n} c_k^2 \\ &= \|w\|^2 - \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n}(a_k - c_k)^2 \\ &= \|w\|^2 - \|u\|^2 + \sum_{k=1}^{n}(a_k - c_k)^2.\end{aligned}$$

It follows that

$$\|w-v\|^2 \geq \|w\|^2 - \|u\|^2$$

with equality if and only if $\sum_{k=1}^{n}(a_k - c_k)^2 = 0$, i.e. if and only if $c_k = a_k = \langle w, v_k\rangle$ for all $k = 1, \ldots, n$. □

Lemma 4.12. *The family of functions*

$$\mathcal{F} = \left\{ \frac{1}{\sqrt{2}},\ \cos(nx),\ \sin(nx)\ :\ n \geq 1 \right\},$$

is orthonormal in $L^2([-\pi,\pi],\mu,\mathbb{R})$.

Proof. We have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{ikx} d\mu = \begin{cases} 0 \text{ if } k \neq 0 \\ 2 \text{ if } k = 0. \end{cases}$$

Use the formulae $\cos(nx) = (e^{inx} + e^{-inx})/2$ and $\sin(nx) = (e^{inx} - e^{-inx})/2i$ to obtain the result. □

Fourier Series

As in Chapter 1, the Fourier series of an integrable function f is

$$\frac{a_0}{\sqrt{2}}\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)d\mu$ and, for $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)d\mu, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)d\mu,\ n \geq 1,$$

where we have written the integrals with respect to μ as we do not assume f is Riemann integrable. (We have writtten the first term $a_0/2$ as $(a_0/\sqrt{2})(1/\sqrt{2})$ for a reason.)

In terms of the inner product, we have

$$\left\langle f, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}}\ d\mu = \frac{a_0}{\sqrt{2}},$$

and, for $n \geq 1$,

$$\langle f, \cos(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) d\mu = a_n, \quad \langle f, \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) d\mu = b_n,$$

the Fourier coefficients of f. Thus the Fourier series may be expressed as

$$\left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \langle f, \cos(nx) \rangle \cos(nx) + \sum_{n=1}^{\infty} \langle f, \sin(nx) \rangle \sin(nx).$$

Define

$$\varphi_n(x) = \begin{cases} \sin(-nx) \text{ if } n < 0 \\ \frac{1}{\sqrt{2}} \text{ if } n = 0 \\ \cos(nx) \text{ if } n > 0. \end{cases}$$

Then the Fourier series has the succinct expression

$$\sum_{n=-\infty}^{\infty} \langle f, \varphi_n \rangle \varphi_n(x).$$

Also,

$$\begin{aligned} S_n(f, x) &= \left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \sum_{k=1}^{n} \langle f, \cos(kx) \rangle \cos(kx) + \sum_{k=1}^{n} \langle f, \sin(kx) \rangle \sin(kx) \\ &= \sum_{k=-n}^{n} \langle f, \varphi_k \rangle \varphi_k(x). \end{aligned}$$

and

$$\sigma_n(f, x) = \frac{1}{n} \left(S_0(f, x) + S_1(f, x) + \cdots + S_{n-1}(f, x) \right) = \sum_{k=-(n-1)}^{n-1} \frac{n - |k|}{n} \langle f, \varphi_k \rangle \varphi_k(x) .$$

Theorem 4.13 (Riesz-Fischer Theorem). *Let $f \in L^2([-\pi, \pi], \mu, \mathbb{R})$. Then $S_n(f, \cdot)$ converges to f in $L^2([-\pi, \pi], \mu, \mathbb{R})$, i.e,*

$$\|S_n(f, \cdot) - f\|_2 = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |S_n(f, \cdot) - f|^2\ d\mu \right)^{1/2} \to 0, \text{ as } n \to +\infty.$$

Before we prove this, we recall Fejér's Theorem (Theorem 1.3) from Chapter 1. Here is a slightly specialized version: *Suppose that* $g : [-\pi, \pi] \to \mathbb{R}$ *is continuous and* $g(-\pi) = g(\pi)$. *Then the sequence of functions* $\sigma_n(g, \cdot)$ *converges uniformly to* g, *as* $n \to +\infty$.

If we define

$$\|\sigma_n(g, \cdot) - g\|_\infty = \sup_{x \in [-\pi,\pi]} |\sigma_n(g, x) - g(x)|$$

then uniform convergence is equivalent to $\lim_{n \to +\infty} \|\sigma_n(g, \cdot) - g\|_\infty = 0$. Also note that

$$\|\sigma_n(g, \cdot) - g\|_2 \leq \sqrt{2} \|\sigma_n(g, \cdot) - g\|_\infty.$$

Proof. Suppose $f \in L^2([-\pi, \pi], \mu, \mathbb{R})$ and let $\epsilon > 0$ be given. By Theorem 4.5, we can find a continuous function $g : [-\pi, \pi] \to \mathbb{R}$ such that $\|f - g\|_2 < \epsilon/2$.

By Fejér's Theorem, we can choose N sufficiently large that

$$n \geq N \quad \Longrightarrow \quad \|\sigma_n(g, \cdot) - g\|_\infty < \frac{\epsilon}{2\sqrt{2}}.$$

Thus, if $n \geq N$ then

$$\|\sigma_n(g, \cdot) - g\|_2 \leq \sqrt{2} \|\sigma_n(g, \cdot) - g\|_\infty < \frac{\epsilon}{2}.$$

Combing the two estimates, if $n \geq N$ then

$$\|f - \sigma_n(g, \cdot)\|_2 \leq \|f - g\|_2 + \|g - \sigma_n(g, \cdot)\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We may write

$$\sigma_n(g, x) = \sum_{k=-(n-1)}^{n-1} d_k \varphi_k(x),$$

for some $d_k \in \mathbb{R}$. By Lemma 4.11,

$$\left\| f - \sum_{k=-(n-1)}^{n-1} \langle f, \varphi_k \rangle \, \varphi_k(x) \right\|_2 \leq \left\| f - \sum_{k=-(n-1)}^{n-1} d_k \, \varphi_k(x) \right\|_2.$$

Thus, if $n \geq N$ then $\|f - S_n(f, \cdot)\|_2 < \epsilon$, as required. □

Having proved the theorem, we are now entitled to say that for $f \in L^2([-\pi, \pi], \mu, \mathbb{R})$,

$$f = \sum_{n=-\infty}^{\infty} \langle f, \varphi_n \rangle \varphi_n \tag{$*$}$$

in $L^2([-\pi, \pi], \mu, \mathbb{R})$.

Theorem 4.14.

$$\mathcal{F} = \left\{ \frac{1}{\sqrt{2}},\ \cos(nx),\ \sin(nx)\ :\ n \geq 1 \right\} = \{\varphi_n\ :\ n \in \mathbb{Z}\}$$

is a (Schauder) basis for the vector space $L^2([-\pi,\pi],\mu,\mathbb{R})$. In other words, for each $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ there is a unique sequence $\{c_n, n \in \mathbb{Z}\}$ such that $f = \sum_{n=-\infty}^{\infty} c_n\varphi_n$ in $L^2([-\pi,\pi],\mu,\mathbb{R})$, that is,

$$\lim_{n\to\infty} \|f - \sum_{k=-n}^{n} c_k\varphi_k\|_2 = 0\,.$$

Consequently, the Fourier series of a function is that function written in terms of the basis.

Proof. The existence of $\{c_n : n \in \mathbb{Z}\}$ follows from $(*)$ above. It remains to show that the representation is unique. Suppose that for some $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ we have $\{c_n, n \in \mathbb{Z}\}$ and $\{d_n, n \in \mathbb{Z}\}$ such that

$$\lim_{n\to\infty} \left\| f - \sum_{k=-n}^{n} c_k\varphi_k \right\|_2 = 0\,, \qquad \lim_{n\to\infty} \left\| f - \sum_{k=-n}^{n} d_k\varphi_k \right\|_2 = 0\,.$$

Then

$$\lim_{n\to\infty} \left\| \sum_{k=-n}^{n} d_k\varphi_k - \sum_{k=-n}^{n} c_k\varphi_k \right\|_2 = \lim_{n\to\infty} \left\| \sum_{k=-n}^{n} (d_k - c_k)\,\varphi_k \right\|_2 = 0\,.$$

Consider any $m \in \mathbb{Z}$ and choose $n \geq |m|$. Then

$$\left\langle \varphi_m, \sum_{k=-n}^{n} (d_k - c_k)\,\varphi_k \right\rangle = \sum_{k=-n}^{n} (d_k - c_k)\,\langle \varphi_m, \varphi_k \rangle = d_m - c_m.$$

On the other hand,

$$\left| \left\langle \varphi_m, \sum_{k=-n}^{n} (d_k - c_k)\,\varphi_k \right\rangle \right| \leq \|\varphi_m\|_2 \left\| \sum_{k=-n}^{n} (d_k - c_k)\,\varphi_k \right\|_2 \to 0, \quad \text{as } n \to +\infty$$

(using the Cauchy-Schwarz Inequality). Taking these together, we see that $c_m = d_m$, for all $m \in \mathbb{Z}$, so the uniqueness of the representation follows. □

Carleson's Theorem

The Riesz-Fischer Theorem was proved in 1907. As it deals with convergence with respect to $\|\cdot\|_2$, it leaves open the question of pointwise convergence, of $S_n(f,x)$ to $f(x)$. We have already seen (Theorem 1.4) that even for continuous f, we do not necessarily have convergence at every point. But what about *almost every* point? It turns out that the answer to this is *yes* for square integrable functions. This was proved by Lennart Carleson in 1966 and is regarded as one of the high points of mathematical analysis in the twentieth century.

Theorem 4.15 (Carleson's Theorem). *Let $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$. Then $S_n(f,x)$ converges to $f(x)$ for μ-a.e. $x \in [-\pi,\pi]$, as $n \to +\infty$.*

Remark. In contrast, the result is *false* if one only assumes that f is integrable. Indeed, there is an example of Kolmogorov (1923) which shows that there is an integrable function $f : [-\pi,\pi] \to \mathbb{R}$ for which $S_n(f,x)$ does not converge at any point.