MATH31011/MATH41011/MATH61011: FOURIER SERIES AND LEBESGUE INTEGRATION

Chapter 3: Measure and Lebesgue Integration

Revision of Riemann Integration

One of the most important operations in mathematics is *integration*. The first rigorous treatment of integration was given by Riemann, as explained in Level 2 Real Analysis. Let us recall the essentials.

Let $[a,b] \subset \mathbb{R}$ be a closed interval and suppose we are given a bounded function $f : [a,b] \to \mathbb{R}$. (For the moment, we impose no other conditions on f.)

A *partition* Δ of $[a,b]$ is a finite set of points $\Delta = \{x_0, x_1, x_2, \dots, x_n\}$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

In other words, we are dividing $[a,b]$ up into subintervals.

We then form the upper and lower Riemann sums

$$U(f,\Delta) = \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x)\ (x_{i+1} - x_i),$$

$$L(f,\Delta) = \sum_{i=0}^{n-1} \inf_{x \in [x_i, x_{i+1}]} f(x)\ (x_{i+1} - x_i).$$

The idea is then that if we make the subintervals in the partition small, these sums will be a good approximation to (our intuitive notion of) the integral of f over $[a,b]$. More precisely, if

$$\inf_{\Delta} U(f,\Delta) = \sup_{\Delta} L(f,\Delta),$$

where the infimum and supremum are taken over all possible partitions of $[a,b]$, then we write

$$\int_a^b f(x)\ dx$$

for their common value and call it the (Riemann) integral of f between those limits. We also say that f is *Riemann integrable*.

The class of Riemann integrable functions includes continuous functions and step functions.

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However, there are many functions for which one wishes to define an integral but which are not Riemann integrable, making the theory rather unsatisfactory. For example, define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ otherwise .} \end{cases}$$

Since between any two distinct real numbers we can find both a rational number and an irrational number, given $0 \leq y < z \leq 1$, we can find $y < x < z$ with $f(x) = 1$ and $y < x' < z$ with $f(x') = 0$. Hence for any partition $\Delta = \{x_0, x_1, \ldots, x_n\}$ of $[0, 1]$, we have

$$\begin{aligned} U(f, \Delta) &= \sum_{i=0}^{n-1} (x_{i+1} - x_i) = 1, \\ L(f, \Delta) &= 0. \end{aligned}$$

Taking the infimum and supremum, respectively, over all partitions Δ, shows that f is *not* Riemann integrable. Fortunately, a far superior theory of integration was devised by Henri Lebesgue in the early part of the twentieth century.

Why does Riemann integration not work for the above function and how could we go about improving it? Let us look again at (and slightly rewrite) the formulae for $U(f, \Delta)$ and $L(f, \Delta)$. We have

$$U(f, \Delta) = \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x)\ l([x_i, x_{i+1}])$$

and

$$L(f, \Delta) = \sum_{i=0}^{n-1} \inf_{x \in [x_i, x_{i+1}]} f(x)\ l([x_i, x_{i+1}]),$$

where, for an interval $[y, z]$,

$$l([y, z]) = z - y$$

denotes its *length*. In the example above, things didn't work because dividing $[0, 1]$ into intervals (no matter how small) did not "separate out" the different values that f could take. But suppose we had a notion of "length" that worked for more general sets than intervals. Then perhaps we could do better by considering more complicated "partitions" of $[0, 1]$, where by partition we now mean a collection of subsets $\{E_1, \ldots, E_m\}$ of $[0, 1]$ such that $E_i \cap E_j = \emptyset$, if $i \neq j$, and $\bigcup_{i=1}^m E_i = [0, 1]$.

In the example, for instance, it might be reasonable to write

$$\begin{aligned} \int_0^1 f(x)\ dx &= 1 \times l([0, 1] \cap \mathbb{Q}) + 0 \times l([0, 1] \backslash \mathbb{Q}) \\ &= l([0, 1] \cap \mathbb{Q}). \end{aligned}$$

It was Lebesgue's idea to introduce the idea of "generalized length" in a rigorous way. The object he constructed is now called Lebesgue measure: it assigns a *size* to (a large class of) subsets of $\mathbb{R}$ and, for intervals, this size is just their length.

We would like to generalize the notion of the length of an interval to a quantity called the Lebesgue measure of a set, which we will denote by μ. More precisely, we would like would be to define, for every $E \subset \mathbb{R}$, a number $\mu(E) \geq 0$ (possibly equal to $+\infty$) satisfying the following natural conditions:

(I) if I is an interval then $\mu(I) = l(I)$ (length property);
(II) for every $x \in \mathbb{R}$, $\mu(x+E) = \mu(E)$, where $x + E = \{x + y : y \in E\}$ (translation invariance);
(III) $E \subset F \implies \mu(E) \leq \mu(F)$ (monotonicity);
(IV) if $\{E_j\}$ is a *countable* collection of disjoint sets (i.e., $E_i \cap E_j = \emptyset$ for $i \neq j$) then

$$\mu\left(\bigcup_j E_j\right) = \sum_j \mu(E_j)$$

(countable additivity).

Remark. We shall frequently need to consider countable collections. Recall that countable means finite or countably infinite (i.e. in one-to-one correspondence with $\mathbb{N}$). Sometimes (as in (IV) above) we shall leave the indexing set unspecified but sometimes we shall let the index run from 1 to ∞. In this latter case, any proof we give will also work for finite collections (with a trivial modification of the argument).

It will turn out that to have (I)-(IV) holding for *all* subsets of $\mathbb{R}$ is too much to ask for: ultimately we will have to restrict to a collection of sets called *measurable sets*.
An important class of sets, which will turn out to be measurable sets, are called *null sets* and are defined as follows.

Definition A set $E \subset \mathbb{R}$ is called a *null set* if, for every $\epsilon > 0$, there is a countable collection of open intervals $\{I_j\}$ such that

$$E \subset \bigcup_j I_j \quad \text{and} \quad \sum_j l(I_j) < \epsilon.$$

We shall use $\mathcal{N}$ to denote the collection of all null sets in $\mathbb{R}$.

Proposition 3.1. *Any countable set is a null set.*

Proof. Exercise. $\square$

Null sets do not have to be countable, as the following example shows.

Proposition 3.2. *The Middle Third Cantor set C is a null set.*

Proof. Recall the construction of the Middle Third Cantor set in Chapter 2. For any $n \geq 1$, $C \subset C_n$ and C_n is a disjoint union of 2^n intervals, each of length 3^{-n}. Call these intervals $C_n^1, \ldots, C_n^{2^n}$. The definition of null set requires open intervals so we enlarge these to make open intervals by setting

$$U_n^i = \left(a - \frac{\delta^n}{2}, b + \frac{\delta^n}{2}\right),$$

where $C_n^i = [a, b]$ and where $\delta > 0$ is closen so small that

$$l(U_n^i) = 3^{-n} + \delta^n < \left(\frac{1}{3} + \delta\right)^n < \frac{1}{2^n}.$$

(We just need to take $\delta < 1/2 - 1/3 = 1/6$.) Then $C \subset \bigcup_{i=1}^{2^n} U_n^i$ and

$$\sum_{i=1}^{2^n} l(U_n^i) < 2^n \left(\frac{1}{3} + \delta\right)^n.$$

Given $\epsilon > 0$, since $2\left(\frac{1}{3} + \delta\right) < 1$, we can choose n sufficiently large that

$$2^n \left(\frac{1}{3} + \delta\right)^n < \epsilon.$$

Thus C is a null set. $\square$

σ-Algebras of Sets

We shall be working with collections of sets and we want our collections to behave sensibly when we take countable unions, countable intersections and complements. To formalise this, we shall introduce the notion of a σ-algebra of sets, working in the general setting of subsets of an arbitrary set X.

Definition A collection $\mathcal{B}$ of subsets of a set X is called a *σ-algebra* if

(i) $\emptyset \in \mathcal{B}$
(ii) $\mathcal{B}$ is closed under complements: $E \in \mathcal{B} \implies X \backslash E \in \mathcal{B}$
(iii) $\mathcal{B}$ is closed under *countable* unions: $\{E_j\}_{j=1}^{\infty} \subset \mathcal{B} \implies \bigcup_{j=1}^{\infty} E_j \in \mathcal{B}$.

(If, in (iii), we replaced "countable" by "finite" then $\mathcal{B}$ would just be called an algebra.)

Simple Examples.

(i) $\mathcal{B} = \{\emptyset, X\}$.
(ii) $\mathcal{B} = \mathcal{P}(X)$ (the set of all subsets of X).

Lemma 3.3. *Let $\mathcal{B}$ be a σ-algebra of subsets of X. Then*

(a) *$X \in \mathcal{B}$;*
(b) *$\mathcal{B}$ is closed under countable intersections: $\{E_j\}_{j=1}^{\infty} \subset \mathcal{B} \implies \bigcap_{j=1}^{\infty} E_j \in \mathcal{B}$.*

Proof.
(a) $\emptyset \in \mathcal{B} \implies X \backslash \emptyset = X \in \mathcal{B}$.
(b) Since $E_j \in \mathcal{B}$, $X \backslash E_j \in \mathcal{B}$, $j \geq 1$. Hence $X \backslash \bigcap_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} X \backslash E_j \in \mathcal{B}$ and so $\bigcap_{j=1}^{\infty} E_j \in \mathcal{B}$. $\square$

Lemma 3.4. *If $\{\mathcal{A}_\alpha\}_{\alpha\in\mathcal{I}}$ is an arbitrary collection of σ-algebras of subsets of X then*

$$\bigcap_{\alpha\in\mathcal{I}} \mathcal{A}_\alpha$$

is also a σ-algebra.

Proof. Exercise. $\square$

Let $\mathcal{C}$ be a collection of subsets of X. (We do not assume that $\mathcal{C}$ is a σ-algebra.) A consequence of this lemma is that it makes sense to speak of the *smallest* σ-algebra containing $\mathcal{C}$: this is just the intersection of all σ-algebras which contain $\mathcal{C}$. We also refer to this as the σ-algebra *generated by* $\mathcal{C}$.

Definition Let $\mathcal{O}$ denote the collection of all open intervals in $\mathbb{R}$. The smallest σ-algebra containing $\mathcal{O}$ is called the *Borel σ-algebra* of $\mathbb{R}$. We use $\mathcal{B}$ to denote the Borel σ-algebra.

Measurable Sets

Definition Let $\mathcal{M}$ denote the smallest σ-algebra containing $\mathcal{O}\cup\mathcal{N}$. Sets in $\mathcal{M}$ are called (Lebesgue) measurable. If I is an interval then $\mathcal{M}(I)$ will denote the measurable subsets of I.

Lemma 3.5. *The Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ is contained in $\mathcal{M}$.*

Proof. By definition, if $\mathcal{C}$ is any σ-algebra containing $\mathcal{O}$ then $\mathcal{B}\subset\mathcal{C}$. Now $\mathcal{M}$ is a σ-algebra containing $\mathcal{O}\cup\mathcal{N}$, so, in particular, it is a σ-algebra containing $\mathcal{O}$. Thus $\mathcal{B}\subset\mathcal{M}$. $\square$

We now state a theorem, the proof of which is the main content of the extra material for MATH41011/MATH61011. It asserts the existence of a generalized length satisfying our conditions (I)-(IV), provided we restrict to sets in $\mathcal{M}$.

Theorem 3.6. *There exists a unique function $\mu:\mathcal{M}\to\mathbb{R}^+\cup\{+\infty\}$ such that*

(i) *if I is an interval then $\mu(I)=l(I)$ (length property);*
(ii) *for every $x\in\mathbb{R}$, $\mu(x+E)=\mu(E)$ (translation invariance);*
(iii) *if $\{E_j\}_{j=1}^\infty$ is a countable collection of disjoint sets (in $\mathcal{M}$) then*

$$\mu\left(\bigcup_{j=1}^\infty E_j\right)=\sum_{j=1}^\infty \mu(E_j)$$

(countable additivity);
(iv) *$E\subset F \implies \mu(E)\le\mu(F)$ (montonicity);*
(v) *if $A\in\mathcal{N}$ then $\mu(A)=0$ and, conversely, if $\mu(A)=0$ then $A\in\mathcal{N}$ (null sets property);*
(vi) *if $A\in\mathcal{M}$ then $\mu(A)=\inf\{\mu(U) : U$ is open and $A\subset U\}$ (regularity).*

Proof. Extra reading for MATH41011/MATH61011. $\square$

Lest one thinks that every set in $\mathbb{R}$ is in $\mathcal{M}$, we have the following.

Theorem 3.7. *There is a subset set of $\mathbb{R}$ which is not contained in $\mathcal{M}$, i.e., $\mathcal{M}\neq\mathcal{P}(\mathbb{R})$.*

Proof. Extra reading for MATH41011/MATH61011. $\square$

We shall develop our theory of integration over the interval $[0, 1]$; any other interval $[a, b]$ – or even the whole real line $\mathbb{R} = (-\infty, +\infty)$ – may be treated in exactly the same way. (In Chapter 4, we will integrate over $[-\pi, \pi]$ because that is convenient for Fourier series.) Indeed, we could equally well develop the theory over any measurable set without any extra effort.

For simplicity of notation, we shall write our integrals (once we've defined them) simply as $\int f d\mu$, where the region of integration is understood. If we wanted to be more precise, we could write $\int_0^1 f d\mu$ or, for the interval $[a, b]$, $\int_a^b f d\mu$. If we were dealing with an arbitrary measurable set $E \in \mathcal{M}$, then the standard notation is $\int_E f d\mu$.

We will look at measurable sets contained in our chosen interval and write

$$\mathcal{M}([0, 1]) = \{A \subset [0, 1] : A \in \mathcal{M}\}.$$

(More generally, we could consider $\mathcal{M}([a, b]) = \{A \subset [a, b] : A \in \mathcal{M}\}$ or $\mathcal{M}(E) = \{A \subset E : A \in \mathcal{M}\}$.) Note that these collections are σ-algebras by the following

Exercise: Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{R}$ and let $X \in \mathcal{A}$. Let $\mathcal{A}(X)$ denote the collection of sets in $\mathcal{A}$ which are subsets of X. Show that $\mathcal{A}(X)$ is a σ-algebra of subsets of X.

In this section we introduce a class of measurable functions for which it is easy to define the integral.

Definition For a set $E \subset \mathbb{R}$, we define its *characteristic function* (or *indicator function*) $\chi_E : \mathbb{R} \to \mathbb{R}$ by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Definition A function $f : [0, 1] \to \mathbb{R}$ is called a simple function if there are finitely many disjoint sets $A_1, \ldots, A_n \in \mathcal{M}([0, 1])$, such that $\bigcup_{i=1}^n A_i = [0, 1]$, and numbers $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that

$$f = \sum_{i=1}^n \alpha_i \; \chi_{A_i}.$$

Lemma 3.8. *If f and g are simple functions and $k \in \mathbb{R}$ then kf, $|f|$, $f + g$ and fg are simple functions.*

Proof. Exercise.

Definition Let $f : [0, 1] \to \mathbb{R}$ be a simple function given by the formula

$$f = \sum_{i=1}^n \alpha_i \; \chi_{A_i}.$$

Then we make the definition

$$\int f \; d\mu = \sum_{i=1}^n \alpha_i \mu(A_i).$$

This definition satisfies natural properties of an integral:

Lemma 3.9. *If $f, g : [0,1] \to \mathbb{R}$ are simple functions and $a, b \in \mathbb{R}$ then*

(i)
$$\int (af + bg)\ d\mu = a \int f\ d\mu + b \int g\ d\mu;$$

(ii) *if $f(x) \le g(x)$ for all $x \in [0,1]$ then*
$$\int f\ d\mu \le \int g\ d\mu;$$

(iii)
$$\left| \int f\ d\mu \right| \le \int |f|\ d\mu.$$

Proof. Exercise. $\square$

Measurable Functions

We now want to consider more general functions on $[0,1]$. To do things in the greatest generality, we want to allow our functions to take infinite values. For example, we might want to integrate (if we can) functions like $1/x$ or $1/\sqrt{x}$ which are infinite at $x = 0$. (In the Riemann theory, $\int_0^1 x^{-1/2} dx$ is defined to be $\lim_{\epsilon \to 0+} \int_\epsilon^1 x^{-1/2} dx$ but we want to avoid this extra complication.) To allow these infinite values, we introduce the *extended real numbers*
$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}.$$
We often write just ∞ for $+\infty$ and we make the natural definitions like $y + \infty = \infty$, $y - \infty = -\infty$ for any $y \in \mathbb{R}$ and $y \cdot \infty = \infty$ for any $y > 0$, but we leave $\infty - \infty$ and $0 \cdot \infty$ undefined. We write $[a, \infty]$ for $[a, \infty) \cup \{+\infty\}$ etc., and we extend the notions of $\sup(E)$, $\inf(E)$ for $E \subset \mathbb{R}^*$ in the obvious way.

We need to define a class of functions, larger than just simple functions, that are sufficiently regular (with respect to the Lebesgue measurable sets) to make integration possible. These will be called measurable functions and we shall define them after the next lemma.

The proof of the next result will use the following two basic facts about inverse images: Let $f : X \to Y$ be a function between any two sets X and Y. Then

(i) for any sets $E_1, E_2, E_3, \ldots \subset Y$,
$$f^{-1}\left(\bigcup_{n=1}^{\infty} E_n \right) = \bigcup_{n=1}^{\infty} f^{-1}(E_n);$$

(ii) for any set $E \subset Y$, $f^{-1}(Y \setminus E) = X \setminus (f^{-1}(E))$.

Lemma 3.10. *For a function $f : [0,1] \to \mathbb{R}^*$, the following statements are equivalent:*

(i) $f^{-1}([-\infty, a]) = \{x \in [0,1] : f(x) \le a\} \in \mathcal{M}([0,1]) \quad \forall a \in \mathbb{R}$*;*
(ii) $f^{-1}([-\infty, a)) = \{x \in [0,1] : f(x) < a\} \in \mathcal{M}([0,1]) \quad \forall a \in \mathbb{R}$*;*
(iii) $f^{-1}([a, \infty]) = \{x \in [0,1] : f(x) \ge a\} \in \mathcal{M}([0,1]) \quad \forall a \in \mathbb{R}$*;*
(iv) $f^{-1}((a, \infty]) = \{x \in [0,1] : f(x) > a\} \in \mathcal{M}([0,1]) \quad \forall a \in \mathbb{R}$.

Proof. We shall show that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (i).
Assume (i) holds. We have

$$f^{-1}([-\infty, a)) = f^{-1}\left(\bigcup_{n=1}^{\infty}\left[-\infty, a - \frac{1}{n}\right]\right) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a - \frac{1}{n}\right]\right).$$

By (i), $f^{-1}([-\infty, a - n^{-1}]) \in \mathcal{M}([0,1])$ for each n. Since $\mathcal{M}([0,1])$ is a σ-algebra, $f^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, a - n^{-1}]) \in \mathcal{M}([0,1])$. Thus (ii) holds.
Assume (ii) holds. Note that $[a, \infty] = \mathbb{R}^* \setminus [-\infty, a)$. Thus

$$f^{-1}([a,\infty]) = f^{-1}(\mathbb{R}^* \setminus [-\infty, a)) = [0.1] \setminus f^{-1}([-\infty, a)),$$

which is in $\mathcal{M}([0,1])$ by (ii) and the fact that σ-algebras are closed under complements. Thus (iii) holds.
Assume (iii) holds. We have

$$f^{-1}((a,\infty]) = f^{-1}\left(\bigcup_{n=1}^{\infty}\left[a + \frac{1}{n}, \infty\right]\right) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[a + \frac{1}{n}, \infty\right]\right).$$

By (iii), $f^{-1}([a+n^{-1}, \infty]) \in \mathcal{M}([0,1])$ for each n. Since $\mathcal{M}([0,1])$ is a σ-algebra, $f^{-1}((a,\infty]) = \bigcup_{n=1}^{\infty} f^{-1}([a+n^{-1}, \infty]) \in \mathcal{M}([0,1])$. Thus (iv) holds.
Assume (iv) holds. Note that $[-\infty, a] = \mathbb{R}^* \setminus (a, \infty]$. Thus

$$f^{-1}([-\infty, a]) = f^{-1}(\mathbb{R}^* \setminus (a,\infty]) = [0,1] \setminus f^{-1}((a,\infty]),$$

which is in $\mathcal{M}([0,1])$ by (ii) and the fact that σ-algebras are closed under complements. Thus (i) holds. $\square$

Definition A function $f : [0,1] \to \mathbb{R}^*$ is called a *measurable function* if it satisfies one (and hence all) of the conditions in Lemma 3.10.

(More generally, for any measurable set E Lemma 3.10 holds with E in place of $[0,1]$, and we can define that $f : E \to \mathbb{R}^*$ is a measurable function analogously.)

An obvious question now is: which functions are measurable?

Lemma 3.11.
(i) *Simple functions are measurable.*
(ii) *Continuous functions are measurable.*

Proof. Exercise. $\square$

Definition Let $P(x)$ be a statement which may or may not hold for points x. If the set $\{x : P(x) \text{ is false}\}$ is a null set then we say that P holds *almost everywhere* or, more formally, μ-almost everywhere. We usually abbreviate this to μ-a.e..
For example, let $\chi_{\mathbb{Q}}$ be the characteristic function of the rational numbers. Then $\chi_{\mathbb{Q}}(x) = 1$ if $x \in \mathbb{Q}$ and is zero otherwise. Since $\mathbb{Q}$ is a null set, we may say that $\chi_{\mathbb{Q}} = 0$ almost everywhere.

Lemma 3.12.
(i) *Suppose that* $f : [0,1] \to \mathbb{R}^*$ *is a function such that* $f = 0$ *almost everywhere. Then* f *is measurable.*
(ii) *Suppose that* $f, g : [0,1] \to \mathbb{R}^*$ *are two functions such that* $f = g$ *almost everywhere. Then* f *is measurable if and only if* g *is measurable.*

Proof.
(i) To show that f is measurable, we will check criterion (i) from Lemma 3.10. Let $a \in \mathbb{R}$ and consider $f^{-1}([-\infty, a])$. If $a < 0$ then, since $f = 0$ almost everywhere, $f^{-1}([-\infty, a])$ is a null set and so $f^{-1}([-\infty, a]) \in \mathcal{M}([0,1])$. If $a \geq 0$ then, again since $f = 0$ almost everywhere, $(f^{-1}([-\infty, a]))^c$ is a null set, giving $(f^{-1}([-\infty, a]))^c \in \mathcal{M}([0,1])$. Since $\mathcal{M}([0,1])$ is a σ-algebra, this implies that $f^{-1}([-\infty, a]) \in \mathcal{M}([0,1])$. Thus f is measurable.
(ii) Let $A = \{x \in [0,1] : f(x) \neq g(x)\}$. Clearly, we only need prove one implication. Suppose that f is measurable. For $a \in \mathbb{R}$, we have

$$g^{-1}([-\infty, a]) = (g^{-1}([-\infty, a]) \cap A) \cup (g^{-1}([-\infty, a]) \cap A^c).$$

Since A is a null set, $g^{-1}([-\infty, a]) \cap A$ is also a null set and so $g^{-1}([-\infty, a]) \cap A \in \mathcal{M}([0,1])$. On the other hand, if $x \in A^c$ then $f(x) = g(x)$, so

$$g^{-1}([-\infty, a]) \cap A^c = f^{-1}([-\infty, a]) \cap A^c.$$

Now, $f^{-1}([-\infty, a]) \in \mathcal{M}([0,1])$ because f is measurable and $A^c \in \mathcal{M}([0,1])$ because A is a null set, giving that their intersection is also in $\mathcal{M}([0,1])$. This shows that $g^{-1}([-\infty, a]) \cap A^c \in \mathcal{M}([0,1])$.
Since $\mathcal{M}([0,1])$ is closed under unions, we have shown that $g^{-1}([-\infty, a]) \in \mathcal{M}([0,1])$. Hence g is measurable. □

Next, we list some of the basic properties of measurable functions.

Lemma 3.13. *Let* $f_n : [0,1] \to \mathbb{R}^*$, $n \geq 1$, *be a sequence of measurable functions. Then*
(i) $\sup_n f_n$ *and* $\inf_n f_n$ *are measurable;*
(ii) $\limsup_{n\to+\infty} f_n$ *and* $\liminf_{n\to+\infty} f_n$ *are measurable.*
Furthermore, if f_n *converges to* f *pointwise, as* $n \to +\infty$ *(i.e.,* $\lim_{n\to+\infty} f_n(x) = f(x)$ $\forall\, x \in [0,1]$*) then*
(iii) f *is measurable.*

Proof. (i) Note that $\sup_n f_n(x) > a$ if and only if $f_n(x) > a$ for some n. Thus

$$\left\{x \in [0,1] : \sup_n f_n(x) > a\right\} = \bigcup_{n=1}^{\infty} \{x \in [0,1] : f_n(x) > a\},$$

so $\sup_n f_n$ is measurable by criterion (iv) from Lemma 3.10. The proof for $\inf_n f_n$ is left as an exercise.
(ii) Write $g_n = \sup_{k\geq n} f_k$. By part (i), g_n is measurable for each $n \geq 1$. Now, since, for each $x \in [0,1]$, $g_n(x)$ is a decreasing sequence,

$$\limsup_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} g_n(x) = \inf_n g_n(x),$$

so applying (i) again gives the result. The proof for $\liminf_{n\to+\infty} f_n$ is left as an exercise.
(iii) By definition, f_n converges pointwise to f if and only if $f = \limsup_{n\to+\infty} f_n = \liminf_{n\to+\infty} f_n = f$, so f is measurable by part (ii). □

Proposition 3.14. *Let $f, g : [0,1] \to \mathbb{R}^*$ be measurable, $c \in \mathbb{R}$. Then cf, $f+g$ and the product fg (if defined) are measurable.*

Proof. First note that the constant function zero is measurable (since it is simple or by Lemma 3.12).
Now suppose that f is measurable and that $c \in \mathbb{R}$. If $c = 0$ then $cf = 0$ and we have just noted this is measurable. If $c \neq 0$ then

$$\{x \in [0,1] : cf(x) \leq a\} = \begin{cases} \{x \in [0,1] : f(x) \leq a/c\} & \text{if } c > 0, \\ \{x \in [0,1] : f(x) \geq a/c\} & \text{if } c < 0. \end{cases}$$

In each case the measurability of cf follows from the measurability of f by Lemma 3.10.
Suppose next that f and g are measurable. Notice that $f(x)+g(x) > a$ if and only if there exists $r \in \mathbb{Q}$ such that $f(x) > r$ and $g(x) > a - r$. Thus

$$\{x \in [0,1] : f(x) + g(x) > a\} = \bigcup_{r\in\mathbb{Q}} \left(\{x \in [0,1] : f(x) > r\} \cap \{x \in [0,1] : g(x) > a - r\}\right)$$

(a countable union). Since f and g are measurable, each set on the RHS lies in $\mathcal{M}([0,1])$ and hence so does their union. Therefore $f + g$ is measurable.
Finally, we shall show that if f and g are measurable then so is fg. First consider f^2. We have

$$\{x \in [0,1] : f^2(x) \leq a\} = \begin{cases} \emptyset \quad \text{if } a < 0 \\ \{x \in [0,1] : f(x) = 0\} \quad \text{if } a = 0 \\ \{x \in [0,1] : -\sqrt{a} \leq f(x) \leq \sqrt{a}\} \quad \text{if } a > 0 \end{cases}.$$

Since f is measurable, all these sets are in $\mathcal{M}([0,1])$, so f^2 is measurable. Now since

$$fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$$

we can apply the preceeding paragraph to conclude that fg is measurable. □

Integrating Non-Negative Measurable Functions

Theorem 3.15. *Let $f : [0,1] \to \mathbb{R}^*$ be a non-negative ($f(x) \geq 0\ \forall\ x \in [0,1]$) measurable function. Then there exists an increasing sequence of non-negative simple functions f_n, $n \geq 1$, such that f_n converges pointwise to f, as $n \to +\infty$.*

Proof. This is not examinable and is relegated to the appendix. □

Let $f : [0,1] \to \mathbb{R}^*$ be a non-negative measurable function. By Theorem 3.15, we can find an increasing sequence of non-negative simple functions f_n which converge pointwise to f as $n \to +\infty$.

Definition With f and f_n as above, we define

$$\int f \; d\mu = \lim_{n\to+\infty} \int f_n \; d\mu.$$

The limit exists because $\int f_n \; d\mu$ is an increasing sequence in $\mathbb{R}^+ \cup \{+\infty\}$. Note that we may have $\int f d\mu = +\infty$.

We need to know that the definition makes sense: we have to check that it is independent of the choice of sequence of simple functions f_n.

Proposition 3.16. *Let f_n, g_n, $n \geq 1$, be two increasing sequences of non-negative simple functions, which converge pointwise to the same measurable function on $[0,1]$, as $n \to \infty$. Then*

$$\lim_{n\to+\infty} \int f_n \; d\mu = \lim_{n\to+\infty} \int g_n \; d\mu.$$

Proof of Proposition 3.16. This is not examinable and is relegated to the appendix. □

Thus $\int f d\mu$ is well-defined for any non-negative measurable function f. The following basic properties are easy to prove.

Lemma 3.17. *If $f, g : [0,1] \to \mathbb{R}^*$ are non-negative measurable functions and $c \in \mathbb{R}^+$ then*

(i)
$$\int (f+g) \; d\mu = \int f d\mu + \int g \; d\mu;$$

(ii)
$$\int cf \; d\mu = c \int f \; d\mu;$$

(iii) *if $f \geq g$, then*
$$\int f \; d\mu \geq \int g \; d\mu.$$

Proof. Exercise. □

Integrating General Measurable Functions

Let $f : [0,1] \to \mathbb{R}^*$ be a measurable function, which we now allow to take both positive and negative values. Define two non-negative functions $f^+ : [0,1] \to \mathbb{R}^*$ and $f^- : [0,1] \to \mathbb{R}^*$ by

$$f^+(x) = \max\{0, f(x)\} \quad \text{and} \quad f^-(x) = \max\{0, -f(x)\}.$$

Then we can write

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$

Lemma 3.18. *If $f : [0,1] \to \mathbb{R}^*$ is measurable then $f^+ : [0,1] \to \mathbb{R}^*$, $f^- : [0,1] \to \mathbb{R}^*$ and $|f| : [0,1] \to \mathbb{R}^*$ are all measurable.*

In particular, $\int f^+ \, d\mu$, $\int f^- \, d\mu$ and $\int |f| \, d\mu$ are all defined but they may be infinite. Because $|f| = f^+ + f^-$, we have

$$\int |f| \, d\mu = \int f^+ \, d\mu + \int f^- \, d\mu.$$

Hence

$$\int |f| \, d\mu < +\infty \iff \int f^+ \, d\mu < +\infty \text{ and } \int f^- \, d\mu < +\infty.$$

Definition If $\int |f| \, d\mu < +\infty$ then we say that f is *integrable.* If f is integrable then we define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

(If f is not integrable then $\int f \, d\mu$ is defined to be $+\infty$ if $\int f^- \, d\mu$ is finite, $-\infty$ if $\int f^+ \, d\mu$ is finite and it is not defined if both $\int f^- \, d\mu$, $\int f^+ \, d\mu$ are infinite.) We shall write $\mathcal{L}^1([0,1],\mu)$ for the set of integrable functions $f : [0,1] \to \mathbb{R}^*$.

Lemma 3.19. *Let $f : [0,1] \to \mathbb{R}^*$ be measurable. Then f is integrable if and only if $|f|$ is integrable. Furthermore, if f is integrable then*

(i) $\left|\int f \, d\mu\right| \le \int |f| \, d\mu$;
(ii) *f is finite valued μ-a.e. on* $[0,1]$.

Proof. The first statement is immediate from the definition of integrability. The proof of the inequality $\left|\int f \, d\mu\right| \le \int |f| \, d\mu$ is left as an exercise. For the final statement, suppose that f is not finite valued μ-a.e. If $E_1 = \{x \in [0,1] : f(x) = +\infty\}$ and $E_2 = \{x \in [0,1] : f(x) = -\infty\}$ then either $\mu(E_1) > 0$ or $\mu(E_2) > 0$ (or both). If $\mu(E_1) > 0$ then, for all $n \ge 1$, $f^+ \ge n\chi_{E_1}$, so that, by Lemma 3.17(iii),

$$\int f^+ \, d\mu \ge \int n\chi_{E_1} \, d\mu = n\mu(E_1).$$

Letting $n \to +\infty$ shows that $\int f^+ \, d\mu = +\infty$, contradicting the integrability of f. If $\mu(E_2) > 0$ then, for all $n \ge 1$, $f^- \ge n\chi_{E_2}$, again giving a contradiction. □

Proposition 3.20. *If f, g are integrable and $c \in \mathbb{R}$ then cf and $f + g$ (if defined) are integrable and furthermore $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$ and $\int cf \, d\mu = c \int f \, d\mu$.*

Proof. Exercise. □

Proposition 3.21. *Suppose that $f, g : [0,1] \to \mathbb{R}^*$ are measurable.*

(i) *If $f = g$ μ-a.e. and f is integrable then g is integrable and $\int f \, d\mu = \int g \, d\mu$.*
(ii) *If f is integrable and $|g| \le |f|$ μ-a.e. then g is integrable.*
(iii) *If f is integrable and $f \ge 0$ μ-a.e., then $\int f \, d\mu = 0$ implies that $f = 0$ μ-a.e. (In particular, if $f > 0$ μ-a.e. then $\int f \, d\mu > 0$.)*
(iv) *If f, g are integrable and $f \ge g$ μ-a.e. then $\int f \, d\mu \ge \int g \, d\mu$.*

Proof. (i) If $f = g$ μ-a.e. then $f^+ = g^+$ μ-a.e. and $f^- = g^-$ μ-a.e. The proof that $\int f^+ \, d\mu = \int g^+ \, d\mu$ and $\int f^- \, d\mu = \int g^- \, d\mu$ follows from the construction of the integral of non-negative measurable functions as limits of integrals of simple functions and, in particular, from an examination of the proof of Theorem 3.15. (More precisely, the sets $Q_{n,k}$ defined in that proof, for f^+ and g^+ respectively, have the same measure.)
(ii) Let $E = \{x \in [0,1] : |g(x)| > |f(x)|\}$, then $E \in \mathcal{M}([0,1])$ and $\mu(E) = 0$. Define a measurable function h by

$$h(x) = \begin{cases} g(x) & \text{if } x \notin E \\ f(x) & \text{if } x \in E \end{cases}.$$

Since $g = h$ μ-a.e., by (i) it suffices to show that h is integrable. However, $|h(x)| \leq |f(x)|$ for all $x \in [0,1]$, so

$$\int |h| \, d\mu \leq \int |f| \, d\mu < +\infty,$$

completing the proof.
(iii) By (i), we can change f on a set of measure zero so that $f \geq 0$ everywhere without changing the problem. Note that

$$\{x \in [0,1] : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n,$$

where

$$A_n = \left\{ x \in [0,1] : f(x) > \frac{1}{n} \right\}.$$

Thus if $\mu(\{x \in [0,1] : f(x) > 0\}) > 0$ then there exists $n \geq 1$ such that $\mu(A_n) > 0$. Then $f \geq \frac{1}{n}\chi_{A_n}$, so that

$$\int f \, d\mu \geq \frac{1}{n} \int \chi_{A_n} \, d\mu = \frac{1}{n} \, \mu(A_n) > 0.$$

Hence, if $\int f \, d\mu = 0$, we must have $\mu(\{x \in [0,1] : f(x) > 0\} = 0$, i.e., $f = 0$ μ-a.e.
(iv) By (i), adjusting g on a set of measure zero if necessary, we can assume that $f \geq g$ everywhere, without affecting the value of the integrals. Then $f - g \geq 0$ so

$$\int (f-g) \, d\mu = \int f d\mu - \int g d\mu \geq 0\,.$$

Limit Theorems

In the final section of this chapter we want to say something about how, given a sequence of integrable functions f_n, the integrals $\int f_n \, d\mu$ behave as $n \to +\infty$.
First, we give an example to show that if $\lim_{n\to\infty} f_n = f$ (everywhere), where f is integrable, it does not follow that $\int f_n \, d\mu$ converges to $\int f \, d\mu$.

Example. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^2 & \text{if } 0 < x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, for every $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = 0$. However,

$$\int f_n \; d\mu = n^2 \mu((0, 1/n]) = n$$

which does not converge to 0, as $n \to +\infty$.

However, imposing extra conditions give the following results.

Theorem 3.22 (Monotone Convergence Theorem). *Suppose that f_n, $n \geq 1$, is an increasing sequence of non-negative measurable functions. Write $f = \lim_{n\to+\infty} f_n$. (Since the f_n are increasing, f is well defined but may take infinite values.) Then*

$$\lim_{n\to+\infty} \int f_n \; d\mu = \int f \; d\mu.$$

Proof. For each $n \geq 1$ choose an increasing sequence $f_{n,k}$ of non-negative simple functions converging to f_n, as $k \to +\infty$. Set

$$g_k = \max_{n\leq k} f_{n,k},$$

then g_k is an increasing sequence of non-negative simple functions. Set $g = \lim_{k\to+\infty} g_k$. For $1 \leq n \leq k$,

$$f_{n,k} \leq g_k \leq f_k \leq f, \tag{$*$}$$

so letting $k \to +\infty$ gives, for all $n \geq 1$,

$$f_n \leq g \leq f$$

and letting $n \to +\infty$ gives $f = g$.
Integrating $(*)$ gives that for $1 \leq n \leq k$,

$$\int f_{n,k} \; d\mu \leq \int g_k \; d\mu \leq \int f_k \; d\mu,$$

so letting $k \to +\infty$ gives, for all $n \geq 1$,

$$\int f_n \; d\mu \leq \int g \; d\mu \leq \lim_{k\to+\infty} \int f_k \; d\mu.$$

Now let $n \to +\infty$ to obtain

$$\lim_{n\to+\infty} \int f_n \; d\mu \leq \int g \; d\mu \leq \lim_{k\to+\infty} \int f_k \; d\mu.$$

Since $f = g$, this is the required result. □

In the proof of the next lemma, we will use the following fact: if a_n, $n \geq 1$ is a sequence of real numbers and $a \in \mathbb{R}$ then

$$\liminf_{n\to\infty} (a_n + a) = \liminf_{n\to\infty} a_n + a.$$

Theorem 3.23 (Fatou's Lemma). *Let f_n, $n \geq 1$, be a sequence of measurable functions which is bounded below by an integrable function g. Then*

$$\int \liminf_{n\to+\infty} f_n \; d\mu \leq \liminf_{n\to+\infty} \int f_n \; d\mu.$$

[Note that the integrals are defined since $f_n \geq g$ for all $n \in \mathbb{N}$ implies $f_n^- \leq g^-$ and $(\liminf_{n\to+\infty} f_n)^- \leq g^-$.]

Proof. Let $h_n = f_n - g$, then h_n is non-negative. Set $g_n = \inf_{k\geq n} h_k$, then g_n is an increasing sequence of non-negative measurable functions so, by the Monotone Convergence Theorem,

$$\int \lim_{n\to+\infty} g_n \; d\mu = \lim_{n\to+\infty} \int g_n \; d\mu.$$

Hence

$$\begin{aligned}
\int \liminf_{n\to+\infty} f_n \; d\mu &= \int \left(\lim_{n\to+\infty} g_n + g \right) d\mu \\
&= \lim_{n\to+\infty} \int (g_n + g) \; d\mu \\
&\leq \lim_{n\to+\infty} \left(\inf_{k\geq n} \int (h_k + g) \; d\mu \right) \\
&= \lim_{n\to+\infty} \left(\inf_{k\geq n} \int f_k \; d\mu \right) \\
&= \liminf_{n\to+\infty} \int f_n \; d\mu,
\end{aligned}$$

as required. □

Corollary 3.24. *Let f_n be a sequence of measurable functions bounded above by an integrable function. Then*

$$\limsup_{n\to+\infty} \int f_n \; d\mu \leq \int \limsup_{n\to+\infty} f_n \; d\mu.$$

Proof. Apply Fatou's Lemma to $-f_n$. □

Theorem 3.26 (Dominated Convergence Theorem[1]). *Let f_n, $n \geq 1$, be a sequence of measurable functions such that $f_n(x)$ converges to $f(x)$ for μ-a.e. x. Suppose there exists an integrable function $g \geq 0$ such that, for all $n \geq 1$, $|f_n(x)| \leq g(x)$ for μ-a.e x. Then f_n, f are integrable and*

$$\lim_{n\to+\infty} \int f_n \; d\mu = \int f \; d\mu.$$

[1] Also known as the Lebesgue Convergence Theorem.

Proof. Note that f_n, f are integrable by Proposition 3.21(ii) since $|f_n|, |f| \leq g$ μ a.e. Define modified functions $h_n = f_n\chi_E$ and $h = f\chi_E$, where

$$E = \left\{ x : \lim_{n\to+\infty} f_n(x) = f(x) \text{ and } |f_n(x)| \leq g(x) \text{ for all } n \geq 1 \right\}.$$

Then $h_n = f_n$ μ-a.e. and $h = f$ μ-a.e.. By Proposition 3.21, h, h_n are integrable and $\int f\ d\mu = \int h\ d\mu$, $\int f_n\ d\mu = \int h_n\ d\mu$. Thus it suffices to show that

$$\lim_{n\to+\infty} \int h_n\ d\mu = \int h\ d\mu.$$

Because of the way we have defined h_n and h, we have

$$\lim_{n\to+\infty} h_n(x) = h(x),$$

for all x. Furthermore, $|h_n(x| = |f_n(x)\chi_E(x)| \leq g(x)$ for all x so $|h(x)| \leq g(x)$ for all x. To complete the proof we shall show that

$$\lim_{n\to+\infty} \left| \int h_n\ d\mu - \int h\ d\mu \right| = 0.$$

We have

$$\begin{aligned} \left| \int h_n\ d\mu - \int h\ d\mu \right| &= \left| \int (h_n - h)\ d\mu \right| \\ &\leq \int |h_n - h|\ d\mu, \end{aligned}$$

so it will be enough to show that the latter integral tends to zero.
Note that $\lim_{n\to+\infty} |h_n(x) - h(x)| = 0$ and that

$$0 \leq |h_n(x) - h(x)| \leq |h_n(x)| + |h(x)| \leq 2g(x)$$

for all x, i.e. $|h_n - h|$ is bounded above and below by integrable functions. Applying Fatou's Lemma and its corollary, we have

$$\begin{aligned} \limsup_{n\to+\infty} \int |h_n - h|\ d\mu &\leq \int \limsup_{n\to+\infty} |h_n - h|\ d\mu \\ &= \int 0\ d\mu = 0 \\ &= \int \liminf_{n\to+\infty} |h_n - h|\ d\mu \\ &\leq \liminf_{n\to+\infty} \int |h_n - h|\ d\mu \\ &\leq \limsup_{n\to+\infty} \int |h_n - h|\ d\mu. \end{aligned}$$

Thus, all the terms are equal and we have

$$\lim_{n\to+\infty} \int |h_n - h|\ d\mu = 0,$$

as required. □

Note. For a general measurable $E \subseteq \mathbb{R}$ in place of $[0, 1]$, the (Lebesgue) integral of $f : E \to \mathbb{R}^*$ is defined analogously, and the same results obtained.

We shall end the chapter by justifying our claim that Lebesgue integration is an extension of Riemann integration by showing that if a function is Riemann integrable then it is also Lebesgue integrable and the two integrals agree.

Let $f : [0,1] \to \mathbb{R}$ be bounded. Recall that if $\Delta = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[0,1]$ then we define the lower and upper sums $L(f,\Delta)$ and $U(f,\Delta)$ by

$$L(f,\Delta) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i) \quad \text{and} \quad U(f,\Delta) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i),$$

where

$$m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\} \quad \text{and} \quad M_i = \sup\{f(x) : x \in [x_i, x_{i+1}]\}.$$

The following is a standard result on Riemann integration (although it was not proved in Year 2). It is not really needed for our main result but it will make the proof easier.

Theorem 3.27. *Let Δ_k be a sequence of partitions of $[0,1]$ and let l_k denote the length of the longest interval defined by Δ_k (i.e. if $\Delta_k = \{x_0, x_1, \ldots, x_n\}$ then $l_k = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$). If $\lim_{k\to+\infty} l_k = 0$ and $f : [0,1] \to \mathbb{R}$ is Riemann integrable then*

$$\lim_{k\to+\infty} L(f,\Delta_k) = \lim_{k\to+\infty} U(f,\Delta_k) = \int_0^1 f(x)\,dx.$$

In particular, if Δ_k is the partition $\{i/2^k : i = 0, \ldots, 2^k\}$ then, for any Riemann integrable f,

$$\lim_{k\to+\infty} L(f,\Delta_k) = \lim_{k\to+\infty} U(f,\Delta_k) = \int_0^1 f(x)\,dx.$$

Theorem 3.28. *If a function $f : [0,1] \to \mathbb{R}$ is Riemann integrable then f is integrable in the sense above (which we refer to as being Lebesgue integrable) and*

$$\int f\,d\mu = \int_0^1 f(x)\,dx,$$

where the second integral is the Riemann integral of f.

Proof. Let Δ_k be the partition $\{x_0, x_1, \ldots, x_{2^k}\} = \{i/2^k : i = 0, \ldots, 2^k\}$, as above. Define simple functions f_k, g_k by

$$f_k = \sum_{i=0}^{2^k-1} m_i \chi_{[x_i,x_{i+1})} + f(1)\chi_{\{1\}} \quad \text{and} \quad g_k = \sum_{i=0}^{2^k-1} M_i \chi_{[x_i,x_{i+1}]} + f(1)\chi_{\{1\}},$$

where m_i and M_i are defined above. Clearly, f_k is an increasing sequence of functions, g_k is a decreasing sequence of functions, and $f_k \le f \le g_{k'}$, for all $k, k' \ge 1$.

We have that

$$\int f_k \, d\mu = \sum_{i=0}^{2^k-1} m_i \mu(\,[x_i, x_{i+1})\,) = \sum_{i=0}^{2^k-1} m_i (x_{i+1} - x_i) = L(f, \Delta_k)$$

and

$$\int g_k \, d\mu = \sum_{i=0}^{2^k-1} M_i \mu(\,[x_i, x_{i+1})\,) = \sum_{i=0}^{2^k-1} M_i (x_{i+1} - x_i) = U(f, \Delta_k).$$

Because f is Riemann integrable, Theorem 3.27 gives us that

$$\lim_{k \to +\infty} \int f_k \, d\mu = \lim_{k \to +\infty} L(f, \Delta_k) = \int_0^1 f(x) \, dx.$$

and that

$$\lim_{k \to +\infty} \int g_k \, d\mu = \lim_{k \to +\infty} U(f, \Delta_k) = \int_0^1 f(x) \, dx.$$

Let F denote the pointwise limit of the f_k, which exists because the f_k are increasing. Because $f_k \le g_1$, for all $k \ge 1$,we have $F \le g_1$. By Lemma 3.13, F is measurable and, since $f_1 \le F \le g_1$, F is integrable.
Similarly, let G denote the pointwise limit of the g_k, which exists because the g_k are decreasing. Because $f_1 \le g_k$, for all $k \ge 1$,we have $f_1 \le G$. By Lemma 3.13, G is measurable and, since $f_1 \le G \le g_1$, G is integrable.

We have $f_k \le F \le f \le G \le g_k$ for all k so since f_k, F, g_k, G are integrable (note we do not say anything about f yet since we do not know if it is measurable), also

$$\int f_k \, d\mu \le \int F \, d\mu \le \int G \, d\mu \le \int g_k \, d\mu \,.$$

Taking the limit as k tends to ∞ shows that

$$\int_0^1 f(x) \, dx \le \int F \, d\mu \le \int G \, d\mu \le \int_0^1 f(x) \, dx \,,$$

and so $\int_0^1 f(x) \, dx = \int F \, d\mu = \int G \, d\mu$. Hence $\int (G - F) \, d\mu = 0$. Since $G - F \ge 0$, it follows by Proposition 3.21 that $G - F = 0$ μ-a.e. and as $F \le f \le G$ also $f = F$ μ-a.e. Therefore, again by Proposition 3.21, f is integrable and

$$\int f \, d\mu = \int F \, d\mu = \int_0^1 f(x) \, dx \,.$$

□