Appendix to Chapter 3

THIS MATERIAL IS NOT EXAMINABLE.

Proof of Theorem 3.15

Theorem 3.15. Let \( f : [0, 1] \rightarrow \mathbb{R}^+ \) be a non-negative (\( f(x) \geq 0 \ \forall \ x \in [0, 1] \)) measurable function. Then there exists an increasing sequence of non-negative simple functions \( f_n \), \( n \geq 1 \), such that \( f_n \) converges pointwise to \( f \), as \( n \rightarrow +\infty \).

Proof. For \( n \in \mathbb{N} \) and \( k = 1, 2, \ldots, 2^n \), define

\[
Q_{n,k} = \left\{ x \in [0, 1] : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\}
\]

and write

\[
Q_{n,0} = [0, 1] \setminus \bigcup_{k=1}^{2^n} Q_{n,k} = \{ x \in [0, 1] : f(x) \geq 2^n \}.
\]

Define a simple function \( f_n : [0, 1] \rightarrow \mathbb{R} \) by

\[
f_n = 2^n \chi_{Q_{n,0}} + \sum_{k=1}^{2^n} \frac{k-1}{2^n} \chi_{Q_{n,k}}.
\]

We need to show that \( f_n \) is an increasing sequence and that \( f_n \) converges pointwise to \( f \), as \( n \rightarrow +\infty \).

First we show that \( f_n \) is increasing. If \( x \in [0, 1] \) then either \( x \in Q_{n,0} \) or \( x \in Q_{n,k} \) for some \( k = 1, \ldots, 2^n \).

If \( x \in Q_{n,0} \) then \( f(x) \geq 2^n \) and either \( f(x) \geq 2^{n+1} \) or

\[
\frac{k-1}{2^{n+1}} \leq f(x) < \frac{k}{2^{n+1}},
\]

for some \( 2^{n+1} + 1 \leq k \leq 2^{n+2} \). In the first case, \( f_{n+1}(x) = 2^{n+1} > 2^n = f_n(x) \). In the second case,

\[
f_{n+1}(x) = \frac{k-1}{2^{n+1}} \geq \frac{2^{n+1}}{2^{n+1}} = 2^n = f_n(x).
\]
If $x \in Q_{n,k}$, for $1 \leq k \leq 2^{2n}$, then either
\[
\frac{2k - 2}{2^{n+1}} \leq f(x) < \frac{2k - 1}{2^{n+1}}
\]
so that
\[
f_{n+1}(x) = \frac{2k - 2}{2^{n+1}},
\]
or
\[
\frac{2k - 1}{2^{n+1}} \leq f(x) < \frac{2k}{2^{n+1}}
\]
so that
\[
f_{n+1}(x) = \frac{2k - 1}{2^{n+1}}.
\]
In both cases we have
\[
f_{n+1}(x) \geq \frac{2k - 2}{2^{n+1}} = \frac{k - 1}{2^n} = f_n(x).
\]
Now we show that $f_n \rightarrow f$ pointwise. If $x \in [0,1]$ has $f(x) = +\infty$ then, for every $n$, $f_n(x) = 2^n \rightarrow f(x)$, as $n \rightarrow +\infty$. On the other hand, if $x \in \mathbb{R}$ has $f(x) < +\infty$ then there exists $n_0 \geq 1$ such that $f(x) < 2^n$ for all $n \geq n_0$ and so $x \in Q_{n,k}$ for some $1 \leq k \leq 2^{2n}$ ($n \geq n_0$). Hence
\[
k - \frac{1}{2^n} \leq f(x) < \frac{k}{2^n}
\]
and $f_n(x) = (k - 1)/2^n$, to give an estimate
\[
0 \leq f(x) - f_n(x) < \frac{1}{2^n}, \quad \forall \ n \geq n_0.
\]
Therefore $f_n(x) \rightarrow f(x)$, as $n \rightarrow +\infty$. □

**Proof of Proposition 3.16**

**Proposition 3.16.** Let $f_n, g_n, n \geq 1$, be two increasing sequences of non-negative simple functions, which converge pointwise to the same measurable function on $[0,1]$, as $n \rightarrow \infty$. Then
\[
\lim_{n \rightarrow +\infty} \int f_n \, d\mu = \lim_{n \rightarrow +\infty} \int g_n \, d\mu.
\]

To prove Proposition 3.16 we need the following technical result.

**Lemma 3.16.1.** If $f_n, n \geq 1$, is an increasing sequence of non-negative simple functions and $g$ is a non-negative simple function such that $g \leq \lim_{n \rightarrow \infty} f_n$, then
\[
\int g \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu.
\]

**Proof of Lemma 3.16.1.** We can write $g = \sum_{i=1}^m \alpha_i \chi_{E_i}$, with $\alpha_i \geq 0$ and the $E_i \in \mathcal{M}([0,1])$ disjoint (note $\mu(E_i) \leq 1$), $i = 1, \ldots, m$. Given $\epsilon > 0$, let
\[
B_n = \{ x \in [0,1] : f_n(x) \geq g(x) - \epsilon \}.
\]
Since $f_n$ is increasing, $B_n$ is an increasing sequence of sets and since $\lim_{n \to +\infty} f_n(x) \geq g(x)$ for all $x \in [0, 1]$, we have $\bigcup_{n=1}^{\infty} B_n = [0, 1]$. Hence, for $1 \leq i \leq m$, $E_i = \bigcup_{n=1}^{\infty} (E_i \cap B_n)$ and $\mu(E_i) = \lim_{n \to +\infty} \mu(E_i \cap B_n)$.

Now, on $E_i \cap B_n$, 
\[ f_n(x) \geq g(x) - \epsilon = \alpha_i - \epsilon. \]
Hence, for $x \in [0, 1]$, 
\[ f_n(x) \geq \sum_{i=1}^{m} (\alpha_i - \epsilon) \chi_{E_i \cap B_n}(x) \]
so that 
\[ \int f_n \, d\mu \geq \int \sum_{i=1}^{m} (\alpha_i - \epsilon) \chi_{E_i \cap B_n} \, d\mu. \]
Therefore 
\[ \lim_{n \to +\infty} \int f_n \, d\mu \geq \lim_{n \to +\infty} \int \sum_{i=1}^{m} (\alpha_i - \epsilon) \chi_{E_i \cap B_n} \, d\mu \]
\[ = \lim_{n \to +\infty} \sum_{i=1}^{m} (\alpha_i - \epsilon) \mu(E_i \cap B_n) \]
\[ = \sum_{i=1}^{m} (\alpha_i - \epsilon) \mu(E_i) \]
\[ = \int g \, d\mu - \epsilon \sum_{i=1}^{m} \mu(E_i). \]
Since $\epsilon > 0$ is arbitrary, we can conclude that $\lim_{n \to +\infty} \int f_n \, d\mu \geq \int g \, d\mu$, as required. □

Proof of Proposition 3.16. For each $m \geq 1$, $g_m \leq \lim_{n \to +\infty} g_n = \lim_{n \to +\infty} f_n$, so, by Lemma 3.16.1, 
\[ \int g_m \, d\mu \leq \lim_{n \to +\infty} \int f_n \, d\mu. \]
Hence 
\[ \lim_{m \to +\infty} \int g_m \, d\mu \leq \lim_{n \to +\infty} \int f_n \, d\mu. \]
By symmetry, we also have 
\[ \lim_{n \to +\infty} \int f_n \, d\mu \leq \lim_{m \to +\infty} \int g_m \, d\mu, \]
so the result is proved. □

3