MATH31011/MATH41011/MATH61011: FOURIER SERIES AND LEBESGUE INTEGRATION

Appendix to Chapter 3

THIS MATERIAL IS NOT EXAMINABLE.

Proof of Theorem 3.15

Theorem 3.15. *Let $f : [0,1] \to \mathbb{R}^*$ be a non-negative ($f(x) \geq 0 \ \forall \ x \in [0,1]$) measurable function. Then there exists an increasing sequence of non-negative simple functions f_n, $n \geq 1$, such that f_n converges pointwise to f, as $n \to +\infty$.*

Proof. For $n \in \mathbb{N}$ and $k = 1, 2, \ldots, 2^{2n}$, define

$$Q_{n,k} = \left\{ x \in [0,1] : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\}$$

and write

$$Q_{n,0} = [0,1] \setminus \bigcup_{k=1}^{2^{2n}} Q_{n,k} = \{x \in [0,1] : f(x) \geq 2^n\}.$$

Define a simple function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n = 2^n \chi_{Q_{n,0}} + \sum_{k=1}^{2^{2n}} \frac{k-1}{2^n} \chi_{Q_{n,k}}.$$

We need to show that f_n is an increasing sequence and that f_n converges pointwise to f, as $n \to +\infty$.

First we show that f_n is increasing. If $x \in [0,1]$ then either $x \in Q_{n,0}$ or $x \in Q_{n,k}$ for some $k = 1, \ldots, 2^{2n}$.

If $x \in Q_{n,0}$ then $f(x) \geq 2^n$ and either $f(x) \geq 2^{n+1}$ or

$$\frac{k-1}{2^{n+1}} \leq f(x) < \frac{k}{2^{n+1}},$$

for some $2^{2n+1} + 1 \leq k \leq 2^{2n+2}$. In the first case, $f_{n+1}(x) = 2^{n+1} > 2^n = f_n(x)$. In the second case,

$$f_{n+1}(x) = \frac{k-1}{2^{n+1}} \geq \frac{2^{2n+1}}{2^{n+1}} = 2^n = f_n(x).$$

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If $x \in Q_{n,k}$, for $1 \leq k \leq 2^{2n}$, then either

$$\frac{2k-2}{2^{n+1}} \leq f(x) < \frac{2k-1}{2^{n+1}}$$

so that

$$f_{n+1}(x) = \frac{2k-2}{2^{n+1}},$$

or

$$\frac{2k-1}{2^{n+1}} \leq f(x) < \frac{2k}{2^{n+1}}$$

so that

$$f_{n+1}(x) = \frac{2k-1}{2^{n+1}}.$$

In both cases we have

$$f_{n+1}(x) \geq \frac{2k-2}{2^{n+1}} = \frac{k-1}{2^n} = f_n(x).$$

Now we show that $f_n \to f$ pointwise. If $x \in [0,1]$ has $f(x) = +\infty$ then, for every n, $f_n(x) = 2^n \to f(x)$, as $n \to +\infty$. On the other hand, if $x \in \mathbb{R}$ has $f(x) < +\infty$ then there exists $n_0 \geq 1$ such that $f(x) < 2^n \ \forall\ n \geq n_0$ and so $x \in Q_{n,k}$ for some $1 \leq k \leq 2^{2n}$ $(n \geq n_0)$. Hence

$$\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$$

and $f_n(x) = (k-1)/2^n$, to give an estimate

$$0 \leq f(x) - f_n(x) < \frac{1}{2^n}, \quad \forall\ n \geq n_0.$$

Therefore $f_n(x) \to f(x)$, as $n \to +\infty$. □

Proof of Proposition 3.16

Proposition 3.16. *Let f_n, g_n, $n \geq 1$, be two increasing sequences of non-negative simple functions, which converge pointwise to the same measurable function on $[0,1]$, as $n \to \infty$. Then*

$$\lim_{n\to+\infty} \int f_n \, d\mu = \lim_{n\to+\infty} \int g_n \, d\mu.$$

To prove Proposition 3.16 we need the following technical result.

Lemma 3.16.1. *If f_n, $n \geq 1$, is an increasing sequence of non-negative simple functions and g is a non-negative simple function such that $g \leq \lim_{n\to\infty} f_n$, then*

$$\int g \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu.$$

Proof of Lemma 3.16.1. We can write $g = \sum_{i=1}^{m} \alpha_i \chi_{E_i}$, with $\alpha_i \geq 0$ and the $E_i \in \mathcal{M}([0,1])$ disjoint (note $\mu(E_i) \leq 1$), $i = 1, \ldots, m$. Given $\epsilon > 0$, let

$$B_n = \{x \in [0,1] : f_n(x) \geq g(x) - \epsilon\}.$$

Since f_n is increasing, B_n is an increasing sequence of sets and since $\lim_{n\to+\infty} f_n(x) \geq g(x)$ for all $x \in [0,1]$, we have $\bigcup_{n=1}^{\infty} B_n = [0,1]$. Hence, for $1 \leq i \leq m$, $E_i = \bigcup_{n=1}^{\infty}(E_i \cap B_n)$ and $\mu(E_i) = \lim_{n\to+\infty} \mu(E_i \cap B_n)$.

Now, on $E_i \cap B_n$,

$$f_n(x) \geq g(x) - \epsilon = \alpha_i - \epsilon.$$

Hence, for $x \in [0,1]$,

$$f_n(x) \geq \sum_{i=1}^{m} (\alpha_i - \epsilon)\ \chi_{E_i \cap B_n}(x)$$

so that

$$\int f_n\ d\mu \geq \int \sum_{i=1}^{m} (\alpha_i - \epsilon)\ \chi_{E_i \cap B_n}\ d\mu.$$

Therefore

$$\begin{aligned}
\lim_{n\to+\infty} \int f_n\ d\mu &\geq \lim_{n\to+\infty} \int \sum_{i=1}^{m} (\alpha_i - \epsilon)\ \chi_{E_i \cap B_n}\ d\mu \\
&= \lim_{n\to+\infty} \sum_{i=1}^{m} (\alpha_i - \epsilon)\ \mu(E_i \cap B_n) \\
&= \sum_{i=1}^{m} (\alpha_i - \epsilon)\ \mu(E_i) \\
&= \int g\ d\mu - \epsilon \sum_{i=1}^{m} \mu(E_i).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we can conclude that $\lim_{n\to\infty} \int f_n\ d\mu \geq \int g\ d\mu$, as required. □

Proof of Proposition 3.16. For each $m \geq 1$, $g_m \leq \lim_{n\to+\infty} g_n = \lim_{n\to\infty} f_n$, so, by Lemma 3.16.1,

$$\int g_m\ d\mu \leq \lim_{n\to+\infty} \int f_n\ d\mu.$$

Hence

$$\lim_{m\to+\infty} \int g_m\ d\mu \leq \lim_{n\to+\infty} \int f_n\ d\mu.$$

By symmetry, we also have

$$\lim_{n\to+\infty} \int f_n\ d\mu \leq \lim_{m\to+\infty} \int g_m\ d\mu,$$

so the result is proved. □