MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Chapter 2: Countability and Cantor Sets

Countable and Uncountable Sets

The concept of countability will be important in this course and we shall revise it here. A set E is said to be *countable* if it can be put in one-to-one correspondence with a subset of $\mathbb{N} = \{1, 2, 3, \dots\}$. More mathematically, E is countable if there exists a surjection $f : \mathbb{N} \to E$, or E is empty.[1]
Such sets are either

(a) finite sets; or
(b) countably infinite sets: these can be put into bijection (one-to-one correspondence) with $\mathbb{N}$ itself.

A simpler way to describe countable sets is that their elements can be written as a sequence: $\{x_1, x_2, x_3, \dots\}$. If the set is finite, this is only a finite sequence (going up to x_n, say) but if the set is infinite, it is an infinite sequence.
If a set is not countable then we say it is *uncountable.*
Here are some standard results about countability.

Proposition 2.1.

(i) *Let E be a countable set and let $f : E \to F$ be a surjection. Then F is countable.*
(ii) *Any subset of a countable set is countable.*

Proof. Exercise. □

Proposition 2.2. *$\mathbb{Z}$ is countable.*

Proof. A bijection between $\mathbb{N}$ and $\mathbb{Z}$ can be defined as follows:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \;\cdots \\ 0 & 1 & -1 & 2 & -2 \end{array}$$

□

[1]In what follows this case should often be treated separately but we neglect to do so since it is trivial. You can check that everything still works.

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Proposition 2.3. *If E and F are countable then*

(i) *$E \cup F$ is countable.*
(ii) *$E \times F$ is countable.*

Proof. The cases when one of E, F are empty are trivial. Otherwise there exist surjections $f : \mathbb{N} \to E$ and $g : \mathbb{N} \to F$.

(i) We can define a map $h : \mathbb{N} \to E \cup F$ by

$$h(n) = \quad \begin{cases} f\left(\frac{n}{2}\right) & \text{if n is even} \\ g\left(\frac{n+1}{2}\right) & \text{if n is odd,} \end{cases}$$

and it is easy to check that this is a surjection from $\mathbb{N}$ onto $E \cup F$.

(ii) Define $h : \mathbb{N} \times \mathbb{N} \to E \times F$ by $h(m, n) = (f(m), g(n))$. Again, it is easy to check that this is a surjection. So the result reduces to showing that $\mathbb{N} \times \mathbb{N}$ is countable (since then there is a surjection $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, making $h \circ \phi : \mathbb{N} \to E \times F$ a surjection).
Using the array

$$\begin{array}{ccccc} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

one can construct a bijection $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by "diagonal enumeration": $\phi(1) = (1, 1), \phi(2) = (1, 2), \phi(3) = (2, 1), \phi(4) = (1, 3), \phi(5) = (2, 2), \phi(6) = (3, 1)$, etc. $\square$

Proposition 2.4. *$\mathbb{Q}$ is countable.*

Proof. Consider the array

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

This contains the set $\mathbb{Q}^>$ of all rational numbers > 0, with some duplications (e.g. $1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \cdots$). Defining a surjection from $\mathbb{N}$ onto $\mathbb{Q}^>$ by the diagonal method as above shows that $\mathbb{Q}^>$ is countable. Similarly, $\mathbb{Q}^<$ is countable so since $\mathbb{Q} = \mathbb{Q}^> \cup \mathbb{Q}^< \cup \{0\}$, $\mathbb{Q}$ is countable by Proposition 2.3(i). $\square$

More generally, we have:

Proposition 2.5. *If E_n is countable, $n \in \mathbb{N}$, then $E = \bigcup_{n=1}^{\infty} E_n$ is countable.*

Proof. Omitted.[2] $\square$

[2]Intuitively, we take a surjection $h_n : \mathbb{N} \to E_n$ for each n and note that $g : \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} E_n$ defined by $g(n, m) = h_n(m)$ is a surjection. Hence $\bigcup_{n=1}^{\infty} E_n$ is countable by Proposition 2.1, since $\mathbb{N} \times \mathbb{N}$ is countable by Proposition 2.3. In general the existence of a set of surjections as above can only be guaranteed by some form of the Axiom of Choice. In this course (in common with most areas of mathematics) we assume the Axiom of Choice, hence the result.

Proposition 2.6. $\mathbb{R}$ *is uncountable.*

Proof. Every real number can be represented by a decimal expansion. For simplicity, just consider the numbers in $[0,1)$. Such a number has an expansion

$$0.a_1a_2a_3a_4\ldots$$

where $a_n \in \{0,1,2,\ldots,9\}$. Sometimes this expansion is not unique (e.g. $0.1000\ldots = 0.0999\ldots$) but choosing to represent such numbers by the expansion ending in 000... rather than 999... makes it unique (exercise).

Suppose $[0,1)$ were countable. Then we could list its elements (i.e., put them into one-to-one correspondence with $\mathbb{N}$):

$$\begin{array}{lcl} x_1 & = & 0.a_1^{(1)}a_2^{(1)}a_3^{(1)}a_4^{(1)}\ldots \\ x_2 & = & 0.a_1^{(2)}a_2^{(2)}a_3^{(2)}a_4^{(2)}\ldots \\ x_3 & = & 0.a_1^{(3)}a_2^{(3)}a_3^{(3)}a_4^{(3)}\ldots \\ x_4 & = & 0.a_1^{(4)}a_2^{(4)}a_3^{(4)}a_4^{(4)}\ldots \\ \vdots & & \end{array}$$

But it is possible to construct a number y not in this list: define

$$b_n = \begin{cases} 5 & \text{if } a_n^{(n)} \neq 5 \\ 6 & \text{if } a_n^{(n)} = 5 \end{cases}.$$

Then $y = 0.b_1b_2b_3b_4\ldots$ is not in the above list, since it differs from x_n in the nth decimal place. Therefore $[0,1)$ is uncountable and hence $\mathbb{R}$ is uncountable. $\square$

The Middle Third Cantor Set

We shall now describe a more complicated subset of $\mathbb{R}$ which is uncountable. This set is called the Middle Third Cantor set and we shall return to it later in the course.

Geometric description. We start with the unit interval

$$C_0 = [0,1].$$

Now define a new set

$$C_1 = C_0 \backslash (1/3, 2/3) = [0,1/3] \cup [2/3,1],$$

i.e., we obtain C_1 by deleting the open middle third of C_0.

Next we obtain a new set C_2 by deleting the open middle thirds of each of the intervals making up C_1,

$$C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1].$$

Continue in this way to obtain sets C_n, $n \geq 0$, where C_n consists of 2^n disjoint closed intervals of length 3^{-n}, formed by deleting the middle thirds of the intervals making up C_{n-1}.

The Middle Third Cantor set is defined to be the intersection of these sets:

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Arithmetic description. We also have

$$C = \left\{ x \in [0,1] : x = \sum_{n=1}^{\infty} a_n 3^{-n},\ a_n \in \{0,2\},\ \text{for all } n \geq 1 \right\}.$$

(Since $0 \leq a_n 3^{-n} \leq 2 \cdot 3^{-n}$, the series converge.) Hence we might also describe C as the set of reals with a ternary expansion

$$0.a_1 a_2 \ldots a_n \ldots$$

such that $a_n = 0$ or 2, for all $n \geq 1$.

Lemma 2.7. *Let $a_n, b_n \in \{0,2\}$ for $n \in \mathbb{N}$ and let $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, $y = \sum_{n=1}^{\infty} b_n 3^{-n}$. Suppose there exists at least one $m \in \mathbb{N}$ such that $a_m \neq b_m$. Then $x \neq y$.*

Proof. Let $M = \min\{n \in \mathbb{N} : a_n \neq b_n\}$, so that $a_n = b_n$, for $n < M$. We have

$$x - y = (a_M - b_M)3^{-M} + \sum_{n=M+1}^{\infty} (a_n - b_n)3^{-n}.$$

Recall that, for $\alpha, \beta \in \mathbb{R}$, $|\alpha + \beta| \geq |\alpha| - |\beta|$, so

$$\begin{aligned} |x - y| &\geq |a_M - b_M| 3^{-M} - \left| \sum_{n=M+1}^{\infty} (a_n - b_n)3^{-n} \right| \\ &\geq 2 \cdot 3^{-M} - \sum_{n=M+1}^{\infty} |a_n - b_n| 3^{-n} \\ &\geq 2 \cdot 3^{-M} - \sum_{n=M+1}^{\infty} 2 \cdot 3^{-n} \\ &= 2 \cdot 3^{-M} - 2 \frac{3^{-(M+1)}}{1 - \frac{1}{3}} \\ &= 2 \cdot 3^{-M} - 3^{-M} = 3^{-M} > 0. \end{aligned}$$

Hence $x \neq y$. □

Corollary 2.8. *The formula*

$$\sum_{n=1}^{\infty} a_n 3^{-n} \mapsto (a_n)_{n=1}^{\infty}$$

gives a well-defined bijective map between C and the set S of all sequences $(a_n)_{n=1}^{\infty}$, where $a_n \in \{0,2\}$, for all $n \in \mathbb{N}$.

Theorem 2.9. *The Middle Third Cantor set C is uncountable.*

Proof. S is clearly in bijection with the set $\{0,1\}^{\mathbb{N}}$, which is shown to be uncountable in an exercise. Thus S is uncountable. From Corollary 2.8, it follows that C is uncountable. □

Revision of sup and inf.

For $\emptyset \neq E \subset \mathbb{R}$, we say that $m \in \mathbb{R}$ is an *upper bound* of E if, for all $x \in E$, $x \leq m$. We write $\mathcal{U}(E)$ for the set of all upper bounds of E:

$$\mathcal{U}(E) = \{m \in \mathbb{R} : \forall x \in E,\, x \leq m\}\,.$$

We say that E is *bounded above* if $\mathcal{U}(E) \neq \emptyset$; if $\mathcal{U}(E) = \emptyset$ then we say that E is *unbounded above.*

Similarly, we say that $l \in \mathbb{R}$ is a *lower bound* of E if, for all $x \in E$, $x \geq l$ and define $\mathcal{L}(E)$ to be the set of all lower bounds of E:

$$\mathcal{L}(E) = \{l \in \mathbb{R} : \forall x \in E,\, l \leq x\}\,.$$

We say that E is *bounded below* if $\mathcal{L}(E) \neq \emptyset$; if $\mathcal{L}(E) = \emptyset$ then we say that E is *unbounded below.*

The real numbers $\mathbb{R}$ have the following *completeness property*:

(1) if $\emptyset \neq E \subset \mathbb{R}$, with E bounded above, then $\mathcal{U}(E)$ contains a smallest member, i.e., there exists $m \in \mathcal{U}(E)$ such that, for all $y \in \mathcal{U}(E)$, $m \leq y$. We write $m = \sup E$ (or sometimes l.u.b. E);

(2) if $\emptyset \neq E \subset \mathbb{R}$, with E bounded below, then $\mathcal{L}(E)$ contains a greatest member, i.e., there exists $l \in \mathcal{L}(E)$ such that, for all $y \in \mathcal{L}(E)$, $y \leq l$. We write $l = \inf E$ (or sometimes g.l.b. E).

If E is unbounded above then we write $\sup E = +\infty$ and if E is unbounded below then we write $\inf E = -\infty$.

Proposition 2.10. *Let $E \subset \mathbb{R}$ be bounded above. Then*

$$m = \sup E \qquad \Longleftrightarrow \qquad \begin{cases} m \in \mathcal{U}(E), \text{ and} \\ \forall \epsilon > 0 \;\; \exists x \in E \text{ such that } m - \epsilon < x. \end{cases}$$

Proof. Exercise. □

Proposition 2.11. *Let $E \subset \mathbb{R}$ be bounded below. Then*

$$l = \inf E \qquad \Longleftrightarrow \qquad \begin{cases} l \in \mathcal{L}(E), \text{ and} \\ \forall \epsilon > 0 \;\; \exists x \in E \text{ such that } x < l + \epsilon. \end{cases}$$

Proof. Exercise. □

Proposition 2.12. *Suppose that $\emptyset \neq E \subset F \subset \mathbb{R}$. Then $\sup E \leq \sup F$ and $\inf F \leq \inf E$.*

Proof. We shall prove the inequality for sup; the argument for inf is similar. If F is unbounded above then the inequality is obvious, so we assume that F is bounded above (so $\mathcal{U}(F) \neq \emptyset$). Let $m \in \mathcal{U}(F)$, then, for all $f \in F$, $f \leq m$. Hence, for all $e \in E$, $e \leq m$, i.e., $m \in \mathcal{U}(E)$. So $\mathcal{U}(F) \subset \mathcal{U}(E)$. In particular, the smallest element of $\mathcal{U}(E)$ is $\leq$ the smallest element of $\mathcal{U}(F)$, i.e., $\sup E \leq \sup F$. □

Limsup and Liminf

Let x_n, $n \geq 1$, be a sequence of real numbers: this may or may not converge. However, even if it does not converge we may still define two useful limiting quantities.

Example Consider the sequence

$$x_n = (-1)^n \left(1 - \frac{1}{n}\right).$$

The first few values are

$$0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots .$$

Clearly, this sequence does not converge but there is an obvious sense in which 1 is a "limiting upper bound" and -1 is a "limiting lower bound". We want to make this idea precise.

But first another example.

Example Consider the sequence

$$x_n = (-1)^n \left(1 + \frac{1}{n}\right).$$

The first few values are

$$-2, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, -\frac{6}{5}, \ldots .$$

Again, it is clear that this sequence does not converge. This time 1 and -1 are not upper and lower bounds but there is still a sense in which they represent greatest and least limiting values.

Definition Given a sequence x_n of real numbers, we say that

$$\limsup_{n \to +\infty} x_n = x$$

(with $x \in \mathbb{R}$) if, given $\epsilon > 0$,

(i) there exists $N \in \mathbb{N}$ such that

$$x_n < x + \epsilon \quad \text{for all } n \geq N;$$

and

(ii)

$$x_n > x - \epsilon \quad \text{for infinitely many values of } n \geq 1.$$

We say that

$$\limsup_{n\to+\infty} x_n = +\infty$$

if, given any $M \geq 1$,

$$x_n > M \quad \text{for infinitely many values of } n \geq 1.$$

We say that

$$\limsup_{n\to+\infty} x_n = -\infty$$

if, $\lim_{n\to\infty} x_n = -\infty$, that is, given any $M \geq 1$, there is $N \in \mathbb{N}$

$$x_n < -M \quad \text{for all } n \geq N.$$

Lemma 2.13. *We have*

$$\limsup_{n\to+\infty} x_n = \lim_{n\to+\infty} \sup_{k\geq n} x_k.$$

In particular, the lim sup *of a sequence of real numbers always exists (though it may be* $\pm\infty$*).*

Proof. Exercise. □

Definition Given a sequence x_n of real numbers, we say that

$$\liminf_{n\to+\infty} x_n = x$$

(with $x \in \mathbb{R}$) if given $\epsilon > 0$,

(i) there exists $N \geq 1$ such that

$$x_n > x - \epsilon \quad \text{for all } n \geq N;$$

and

(ii)

$$x_n < x + \epsilon \quad \text{for infinitely many values of } n \geq 1.$$

We say that

$$\liminf_{n\to+\infty} x_n = -\infty$$

if, given any $M \geq 1$,

$$x_n < -M \quad \text{for infinitely many values of } n \geq 1.$$

We say that

$$\liminf_{n\to+\infty} x_n = \infty$$

if $\lim_{n\to\infty} x_n = \infty$, that is, given any $M \geq 1$, there is $N \in \mathbb{N}$

$$x_n > M \quad \text{for all } n \geq N.$$

Lemma 2.14. *We have*

$$\liminf_{n\to+\infty} x_n = \lim_{n\to+\infty} \inf_{k\geq n} x_k.$$

In particular, the lim inf *of a sequence of real numbers always exists (though it may be* $\pm\infty$*).*

Proof. Exercise. □