MATH31011/MATH41011/MATH61011:
Fourier Analysis and Lebesgue Integration

Chapter 1: Fourier Series

A Crisis in Analysis

This course is primarily about functions $f : \mathbb{R} \to \mathbb{R}$ or $f : [a, b] \to \mathbb{R}$, where $[a, b] \subset \mathbb{R}$ is some interval. Until the early 19th century, people thought of functions as smooth curves, in particular having a derivative at each point. Of course, there were functions not like this (e.g. $f(x) = |x|$, which is not differentiable at $x = 0$) but these were considered silly contrived examples: proper functions were things like x^2, $\cos x$, $\sin x$, $\exp x, \ldots$.
In 1807, Joseph Fourier was investigating the problem of finding a mathematical description of the flow of heat in a very long and thin rectangular plate.[1] He supposed

(a) there is no heat loss from either face of the plate;

(b) the two long sides are held at constant temperature zero.

Then heat is applied to one of the short sides and the other short side is supposed to be infinitely far away. The plate is supposed to be infinitely thin (zero thickness), so it may be represented by the region

$$\{(x, y) \in \mathbb{R}^2 : \ -1 \leq x \leq 1, \ y \geq 0\},$$

with $y = 0$ representing the side to which heat is applied. The temperature at the point (x, y) is denoted $z(x, y)$, so condition (b) above becomes

$$z(-1, y) = z(1, y) = 0, \quad \text{for } y > 0. \tag{1.1}$$

Fourier further supposed that the initial temperature along the short side is a known function:

$$z(x, 0) = f(x)$$

[1] This opening section is based on the account in D. Bressoud, *A Radical Approach to Real Analysis*, The Mathematical Association of America, 2007.

and, to simplify matters, that f is even (i.e. $f(-x) = f(x)$). Then describing the (stable) heat flow is equivalent to solving Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

subject to the boundary condition $z(x, 0) = f(x)$.
Fourier started by trying to find a solution in the simple and important case

$$f(x) = 1 \quad \text{for all } -1 \leq x \leq 1. \tag{1.2}$$

There is an immediate problem: (1.1) and (1.2) taken together imply that the solution will have a discontinuity at $(x, y) = (-1, 0)$ and $(x, y) = (1, 0)$. Fourier realized that this ceases to be a problem if one replaced $f(x) = 1$ with $f(x) = \cos(\pi x/2)$ (since $\cos(-\pi/2) = \cos(\pi/2) = 0$) or, more generally, an arbitrary linear combination of cosine functions as shown:

$$f(x) = a_1 \cos\left(\frac{\pi x}{2}\right) + a_2 \cos\left(\frac{3\pi x}{2}\right) + a_3 \cos\left(\frac{5\pi x}{2}\right) + \cdots + a_n \cos\left(\frac{(2n-1)\pi x}{2}\right). \tag{1.3}$$

Indeed, in this case, one obtains a solution

$$\begin{aligned} z(x, y) &= a_1 e^{-\pi y/2} \cos\left(\frac{\pi x}{2}\right) + a_2 e^{-3\pi y/2} \cos\left(\frac{3\pi x}{2}\right) + \\ &\quad a_3 e^{-5\pi y/2} \cos\left(\frac{5\pi x}{2}\right) + \cdots + a_n e^{-(2n-1)\pi y/2} \cos\left(\frac{(2n-1)\pi x}{2}\right). \end{aligned}$$

(How the solution is obtained is not important for what follows.)
So far, so good: one has a solution whenever $f(x)$ can be written in the form (1.3). For such an f, one has $f(-1) = f(1) = 0$, so $f(x) = 1$ does not fall into this category. Now came Fourier's stroke of genius. He *assumed* that, it was possible to write 1 as an *infinite* cosine series so that for any $-1 < x < 1$ (notice the strict inequalities),

$$1 = a_1 \cos\left(\frac{\pi x}{2}\right) + a_2 \cos\left(\frac{3\pi x}{2}\right) + a_3 \cos\left(\frac{5\pi x}{2}\right) + \cdots + a_n \cos\left(\frac{(2n-1)\pi x}{2}\right) + \cdots$$

and then worked out what the coefficients a_i would have to be. As a result of his calculations, he boldly asserted that, for any $-1 < x < 1$,

$$\begin{aligned} 1 &= \frac{4}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \cdots\right] \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos\left(\frac{(2n-1)\pi x}{2}\right). \end{aligned} \tag{1.4}$$

If this is true then it gives a solution

$$z(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}e^{-(2n-1)\pi y/2}\cos\left(\frac{(2n-1)\pi x}{2}\right)$$

for the temperature of the plate, in the case $f(x) = 1$.
Fourier's solution was not universally accepted: an equation like (1.4) seemed to contradict the established view about the nature of functions. Furthermore, Fourier's derivation of (1.4) contained steps which he could not justify. In particular, at one crucial point he assumes that the integral of an infinite series can be obtained by integrating each terms and then summing, i.e.

$$\int\left(\sum_{n=1}^{\infty} g_n(x)\right)dx = \sum_{n=1}^{\infty}\left(\int g_n(x)dx\right). \tag{1.5}$$

On the other hand, the mathematical solution Fourier derived gave the correct physical answer, as was verified by experiment. One aim of the course will be to establish conditions under which (1.5) is valid.
Let us assume Fourier's equation (1.4) is true. Replacing x by $x+2$ in (1.4) changes the sign of all the cosine terms. Thus, (1.4) is equivalent to the statement that, for $1 < x < 3$,

$$-1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\left(\frac{(2n-1)\pi x}{2}\right).$$

Continuing in this way, the series

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\left(\frac{(2n-1)\pi x}{2}\right)$$

is equal to the function

$$f(x) = \begin{cases} 1 & \text{if } 4m-1 < x < 4m+1,\ m \in \mathbb{Z} \\ 0 & \text{if } x = 2m+1,\ m \in \mathbb{Z} \\ -1 & \text{if } 4m+1 < x < 4m+3,\ m \in \mathbb{Z}. \end{cases}$$

(Draw a picture!) This just did not look right to the mathematicians of 1807. In fact, Fourier's solution was correct. However, it took another 22 years until this was proved by David Gustav Lejeune Dirichlet. (He was only 2 years old in 1807, so he cannot be blamed for not coming up with it immediately.) This was a special case of Dirichlet's Theorem, which many of you will have seen in MATH20401/MATH20411.

Theorem 1.1 *(Dirichlet's Theorem) Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is a function such that*

(i) f *has period* 2π*;*

(ii) f *and its derivative* f' *are both piecewise continuous in* $(-\pi, \pi)$.

Then the associated Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

converges to

(a) $f(x)$ *whenever* f *is continuous at* x*;*

(b)

$$\frac{1}{2}\left(\lim_{\xi \to x+} f(\xi) + \lim_{\xi \to x-} f(\xi)\right)$$

whenever f *is not continuous at* x.

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$ *and, for* $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \ n \geq 1.$$

Remark It is important to be clear about exactly what is meant by the convergence stated above. We mean that for each point x, the sequence of numbers

$$\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx))$$

converges to the number $f(x)$ (or $\frac{1}{2}\left(\lim_{\xi \to x+} f(\xi) + \lim_{\xi \to x-} f(\xi)\right)$), as $N \to +\infty$. This is called *pointwise convergence*. Later, we shall discuss other types of convergence for functions.

Complex Exponentials

It is convenient to express Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

as a series of complex exponentials

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

In particular, this saves on the amount of writing we need to do. We'll now see how to do this.
First, recall the formulae

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

and

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right).$$

For each $n \geq 1$, we have

$$\begin{aligned}
a_n \cos(nx) + b_n \sin(nx) &= a_n\left(\frac{e^{inx} + e^{-inx}}{2}\right) - ib_n\left(\frac{e^{inx} - e^{-inx}}{2}\right) \\
&= \left(\frac{a_n - ib_n}{2}\right) e^{inx} + \left(\frac{a_n + ib_n}{2}\right) e^{-inx} \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(\cos(nt) - i\sin(nt))dt\right) e^{inx} \\
&\quad + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(\cos(nt) + i\sin(nt))dt\right) e^{-inx} \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt\right) e^{inx} + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{int}dt\right) e^{-inx}.
\end{aligned}$$

Thus we obtain

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Continuous Functions and Fourier Series

Notice that Dirichlet's Theorem contains the unnatural and inconvenient condition that

the derivative f' is piecewise continuous in $(-\pi, \pi)$.

Indeed, Dirichlet believed that this condition was really unnecessary and that he would be able to prove that the convergence of Fourier series described in Theorem 1.1 would hold for any continuous functions $f : \mathbb{R} \to \mathbb{R}$ with period 2π. But he was wrong!

Theorem 1.2 *(Du Bois-Reymond, 1873) There exist continuous functions $f : \mathbb{R} \to \mathbb{R}$ with period 2π such that*

$$\lim_{N \to +\infty} \left(\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \right) = \infty,$$

when $x = 0$. Hence the Fourier series associated to f does not converge to $f(0)$ at $x = 0$.

Proof Omitted. ■

It turns out that, for continuous functions, we can get pointwise convergence and, even better, uniform convergence if we perform an averaging operation. More precisely, writing

$$S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) = \sum_{k=-n}^{n} c_k e^{ikx},$$

where a_k, b_k and c_k are as above, we define

$$\sigma_n(f, x) = \frac{1}{n} (S_0(f, x) + S_1(f, x) + \cdots + S_{n-1}(f, x)) = \sum_{k=-(n-1)}^{n-1} \frac{n - |k|}{n} c_k e^{ikx}$$

(σ_n is called the Cesàro average of S_n). The idea is that σ_n is likely to be (and is!) better behaved than S_n.

Definition *We say that a sequence of functions $f_n : [a, b] \to \mathbb{R}$ converges uniformly to a function $f : [a, b] \to \mathbb{R}$, as $n \to +\infty$, if for all $\epsilon > 0$ there exists $N \geq 1$ such that, for all $x \in [a, b]$,*

$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \epsilon.$$

Remark Note that uniform convergence is stronger than pointwise convergence at every point $x \in [a, b]$. The formal definition of pointwise convergence of f_n to f at x is that for all $\epsilon > 0$ there exists $N \geq 1$ (depending on both x and ϵ) such that

$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \epsilon.$$

Theorem 1.3 *(Fejér's Theorem) Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is continuous with period* 2π*. Then the sequence of functions* $\sigma_n(f, \cdot)$ *converges uniformly to* f*, as* $n \to +\infty$*.*

The proof is given in an appendix and is not examinable.

Conclusion

It appears from the preceding discussion that Fourier Series are tricky and complicated. We might conclude that they are tools which are useful in applications – because they work – but are best avoided in when doing mathematical analysis. However, a more optimistic view, which will be advanced later in the course is that they appear complicated only because we are looking at them in the wrong setting. Indeed, in Chapter 4 we will define a particular *vector space* of functions in which the functions

$$1, \ \cos(nx), \ \sin(nx), \quad n \geq 1,$$

form a *basis* and in which a Fourier series of a function is just that function written in terms of this basis. To define this space, we need to think carefully about integration (which we have already seen was a stumbling block in Fourier's original approach to the convergence of his series). After some preliminary work in Chapter 2, we will go on to do this in Chapter 3. Finally, in Chapter 4, we will return to Fourier series.