MATH31011/MATH41011/MATH61011: Fourier Analysis and Lebesgue Integration

Appendix to Chapter 1: Proof of Fejér's Theorem

THIS MATERIAL IS NOT EXAMINABLE.

Theorem (Theorem 1.3, Fejér's Theorem) *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous with period 2π. Then the sequence of functions $\sigma_n(f, \cdot)$ converges uniformly to f, as $n \to +\infty$.*

Before we give the proof, we shall study another way of writing $\sigma_n(f, x)$. We have that

$$
\begin{aligned}
\sigma_n(f,x) &= \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n}\, c_k e^{ikx} \\
&= \frac{1}{2\pi} \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} \left(\int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right) e^{ikx} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n}\, e^{ik(x-t)} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(x-t) dt,
\end{aligned}
$$

where

$$K_n(u) = \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} e^{iku}.$$

Making the substitution $u = x - t$, we have

$$\sigma_n(f,x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(x-t) dt = -\frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(x-u) K_n(u) du$$

$$= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-u)K_n(u)du = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)K_n(u)du.$$

Lemma *$K_n(0) = n$ and*

$$K_n(u) = \frac{1}{n}\left(\frac{\sin nu/2}{\sin u/2}\right)^2,$$

for $u \neq 0$.

Proof Remember that

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta.$$

First note that

$$\sum_{k=-(n-1)}^{n-1} (n-|k|)e^{iku} = \left(\sum_{k=0}^{n-1} e^{i(k-(n-1)/2)u}\right)^2$$

as can be seen by multiplying out the square on the right hand side and collecting terms. Hence the result for $u = 0$, and if $u \neq 0$, then

$$\begin{aligned}
\sum_{k=-(n-1)}^{n-1} (n-|k|)e^{iku} &= \left(\sum_{k=0}^{n-1} e^{i(k-(n-1)/2)u}\right)^2 = \left(e^{-i(n-1)u/2}\sum_{k=0}^{n-1} e^{iku}\right)^2 \\
&= \left(e^{-i(n-1)u/2}\frac{1-e^{inu}}{1-e^{iu}}\right)^2 = \left(\frac{e^{-inu/2} - e^{inu/2}}{e^{-iu/2} - e^{iu/2}}\right)^2 \\
&= \left(\frac{\sin nu/2}{\sin u/2}\right)^2.
\end{aligned}$$

■

We now state some important properties of $K_n(u)$.

Lemma

- $K_n(u) \geq 0$;
- *Let* $\delta > 0$. *Then* $K_n(u) \to 0$, *as* $n \to +\infty$, *uniformly for* $\delta \leq |u| \leq \pi$
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(u)du = 1$ *(for all* $n \geq 1$*).*

Proof Part (1) is obvious, since $K_n(u)$ is a square. For part (2), note that, if $\delta \leq |u| \leq \pi$,

$$K_n(u) \leq \frac{1}{n}\left(\frac{1}{\sin u/2}\right)^2 \leq \frac{1}{n}\left(\frac{1}{\sin \delta/2}\right)^2 \to 0,$$

as $n \to +\infty$. To get part (3), note that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(u)du &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} e^{iku} du \\ &= \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iku} du = 1 \end{aligned}$$

(since only the $k = 0$ term makes a contribution). ■

Definition *Let* $X \subset \mathbb{R}$. *We say that a function* $f : X \to \mathbb{R}$ *is uniformly continuous if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$y, z \in X \text{ and } |y - z| < \delta \quad \Longrightarrow \quad |f(y) - f(z)| < \epsilon.$$

We will use the following standard results.

Lemma *If* $f : [a, b] \to \mathbb{R}$ *is continuous then it is uniformly continuous.*

Lemma *If* $f : [a, b] \to \mathbb{R}$ *is continuous then it is bounded, so that*

$$\|f\|_\infty := \sup_{x \in [a,b]} |f(x)| < +\infty.$$

Remark Neither of these results is true for the open interval (a, b). (For example $f(x) = x^{-1}$ is continuous but not uniformly continuous on $(0, 1)$.)

We can now prove Fejér's Theorem.

Proof By the above lemma, the continuous function $f : [-2\pi, 2\pi] \to \mathbb{R}$ is is uniformly continuous. Thus, given $\epsilon > 0$, there exits $\delta > 0$ such that if $z, y \in [-2\pi, 2\pi]$ and $|z - y| < \delta$ then $|f(z) - f(y)| \leq \epsilon/2$. Also, we can find $N \geq 1$ (depending on ϵ and δ) such that $|K_n(u)| \leq \epsilon/4\|f\|_\infty$, for all $\delta \leq |u| \leq \pi$ and $n \geq N$. We have

$$
\begin{aligned}
&|\sigma_n(f, x) - f(x)| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) K_n(u) du - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) K_n(u) du \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (\, f(x-u) - f(x)\,)\, K_n(u) du \right| \\
&\leq \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} (\, f(x-u) - f(x)\,)\, K_n(u) du \right| + \left| \frac{1}{2\pi} \int_{\pi \geq |u| \geq \delta} (\, f(x-u) - f(x)\,)\, K_n(u) du \right| \\
&\leq \frac{\epsilon}{2} \frac{1}{2\pi} \int_{-\delta}^{\delta} |K_n(u)|\, du + 2\|f\|_\infty \frac{1}{2\pi} \int_{\pi \geq |u| \geq \delta} |K_n(u)|\, du \\
&\leq \frac{\epsilon}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(u) du \, + \, 2\|f\|_\infty \frac{1}{2\pi} \int_{\pi \geq |u| \geq \delta} \frac{\epsilon}{4\|f\|_\infty} du \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}
$$

for all $n \geq N$, giving the result. ∎

Remark *Fejér proved this theorem when he was nineteen!*