MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Solution Sheet 7

1. For each $p > 0$ define $g_p : [-\pi, \pi] \to \mathbb{R}$ by

$$g_p(x) = \begin{cases} \left|\frac{x}{\pi}\right|^{-p} & \text{if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}$$

A simple modification of the proof for f_p from Example 4, Sheet 5, shows that g_p is integrable if and only if $p < 1$. Thus the function $f = g_{1/2}$ is integrable. However, $|g_{1/2}|^2 = g_1$, which is not integrable. Thus, $f = g_{1/2}$ is not square integrable.

2. Note that if $\|f\|_2\|g\|_2 = 0$ then $\|f\|_2 = 0$ or $\|g\|_2 = 0$ so $f = 0$ μ-a.e. or $g = 0$ μ-a.e. and we have equality both in (i) and (ii). Assume $\|f\|_2\|g\|_2 \neq 0$.

(i) Let $f_0 = \frac{f}{\|f\|_2}$, $g_0 = \frac{g}{\|g\|_2}$. Noting that $\|f\|_2$ and $\|g\|_2$ are constants and applying the corollary[1] to Lemma 4.1 to f_0, g_0, we get

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \frac{|fg|}{\|f\|_2\|g\|_2}\, d\mu \leq \left(\frac{\|f\|_2^2}{\|f\|_2^2} + \frac{\|g\|_2^2}{\|g\|_2^2}\right) = 2,$$

and rearranging,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |fg|\, d\mu \leq \|f\|_2\|g\|_2,$$

with equality if and only if $|f| = (\|f\|_2/\|g\|_2)|g|$ μ-a.e.. If $|f| = c|g|$ μ-a.e. for an arbitrary c then $|f|^2 = c^2|g|^2$ μ-a.e. and so

$$\|f\|_2^2 = c^2\|g\|_2^2.$$

[1] Alternatively we can apply Lemma 4.1 rather than its corollary to functions $f_0 = \frac{|f|}{\|f\|_2}$, $g_0 = \frac{|g|}{\|g\|_2}$ obtaining again

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \frac{|fg|}{\|f\|_2\|g\|_2}\, d\mu = \frac{2}{\pi}\int_{-\pi}^{\pi} \frac{|f|}{\|f\|_2}\,\frac{|g|}{\|g\|_2} d\mu \leq \left(\frac{\|\,|f|\,\|_2^2}{\|f\|_2^2} + \frac{\|\,|g|\,\|_2^2}{\|g\|_2^2}\right) = 2,$$

(since $\|\,|f|\,\|_2 = \|\,f\,\|_2$, $\|\,|g|\,\|_2 = \|\,g\,\|_2$) with equality if and only if $|f| = (\|f\|_2/\|g\|_2)|g|$ μ-a.e., and argue as above.

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Thus we necessarily have $c = \|f\|_2/\|g\|_2$ and so the condition for equality becomes $|f| = c|g|$ μ-a.e., as required.
(ii) We have

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\mu\right| \le \frac{1}{\pi}\int_{-\pi}^{\pi} |fg|\,d\mu \le \|f\|_2\|g\|_2,$$

by Hölder's inequality. However, this does not give us the correct condition for equality. Applying Lemma 4.1 to functions tf and g with $t = \frac{1}{\|f\|_2^2}\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\mu$ we get

$$\frac{2}{\pi^2}\frac{1}{\|f\|_2^2}\left(\int_{-\pi}^{\pi} fg\,d\mu\right)^2 \le \left(\frac{1}{\|f\|_2^2}\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\mu\right)^2 \|f\|_2^2 + \|g\|_2^2$$

and rearranging gives

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\mu\right| \le \|f\|_2\|g\|_2.$$

Furthermore, we get equality if and only if $tf = g$ μ-a.e., that is,

$$\left(\frac{1}{\|f\|_2^2}\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\mu\right) f \;=\; g \quad \mu\text{-a.e}$$

and, because in this case

$$\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\mu = \pm\|f\|_2\|g\|_2,$$

this simplifies to

$$f = \frac{\|f\|_2}{\|g\|_2}g \;\; \mu\text{-a.e.} \qquad \text{or} \qquad f = -\frac{\|f\|_2}{\|g\|_2}g \;\; \mu\text{-a.e.}$$

As in part (i), we finish by noting that if $f = cg$ for an arbitrary $c \in \mathbb{R}$ then $|f|^2 = c^2|g|^2$ and so $c = \pm\|f\|_2/\|g\|_2$. Thus the condition for equality becomes $f = cg$ μ-a.e., for some $c \in \mathbb{R}$, as required.

3. Suppose that $f : [-\pi, \pi] \to \mathbb{R}$ is square integrable. The constant function $g = 1$ is also square integrable. Therefore, by Hölder's inequality,

$$\int_{-\pi}^{\pi} |f|\,d\mu = \int_{-\pi}^{\pi} |f\cdot 1|\,d\mu \le \pi\|f\|_2\|1\|_2 = \pi\sqrt{2}\|f\|_2 < +\infty.$$

Thus, f is integrable.

4. Suppose that $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$. If $g(x) = x^n$ then we also have $g \in L^2([-\pi,\pi],\mu,\mathbb{R})$. Applying Hölder's inequality,

$$\begin{aligned}\left(\int_{-\pi}^{\pi} |x^n f(x)|\,d\mu\right)^2 &= \left(\int_{-\pi}^{\pi} |fg|\,d\mu\right)^2 \le \pi^2\|f\|_2^2\|g\|_2^2 = \pi^2\|f\|_2^2\frac{1}{\pi}\int_{-\pi}^{\pi} x^{2n}\,d\mu \\ &= \pi^2\|f\|_2^2\frac{1}{(2n+1)\pi}(\pi^{2n+1} - (-\pi)^{2n+1}) \\ &= \pi^2\|f\|_2^2\frac{2\pi^{2n+1}}{(2n+1)\pi} = \frac{2\pi^{2n+1}}{2n+1}\int_{-\pi}^{\pi} |f|^2\,d\mu.\end{aligned}$$

5. By Theorem 4.5, we can find a continuous function $h : [-\pi, \pi] \to \mathbb{R}$ such that $\|f - h\|_2 < \epsilon/2$. Because h is continuous, it is bounded, i.e. there exists $M > 0$ such that $|h(x)| \le M$ for all $x \in [-\pi, \pi]$. For $n \ge 1$, define continuous functions $h_n : [-\pi, \pi] \to \mathbb{R}$ such that

(1) for $0 \le x \le -\pi + 1/n$, $h_n(x)$ is the linear function with $h_n(-\pi) = 0$ and $h_n(-\pi + 1/n) = h(-\pi + 1/n)$;
(2) for $-\pi + 1/n < x < \pi - 1/n$, $h_n(x) = h(x)$;
(3) for $\pi - 1/n \le x \le \pi$, $h_n(x)$ is the linear function with $h_n(\pi - 1/n) = h(\pi - 1/n)$ and $h_n(\pi) = 0$.

Clearly, $h_n(-\pi) = h_n(\pi) = 0$ and $|h_n(x)| \le M$ for all $x \in [-\pi, \pi]$. By Minkowski Inequality, we have

$$\|f - h_n\|_2 \le \|f - h\|_2 + \|h - h_n\|_2 \,.$$

Since $(h - h_n)^2$ is 0 except on $\left[-\pi, -\pi + \frac{1}{n}\right] \cup \left[\pi - \frac{1}{n}, \pi\right]$ where it is less or equal to $4M^2$, we have

$$\|h - h_n\|_2^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} (h - h_n)^2 \, d\mu \le \frac{4M^2}{\pi} \, \mu \left(\left[-\pi, -\pi + \frac{1}{n}\right] \cup \left[\pi - \frac{1}{n}, \pi\right] \right) = \frac{8M^2}{n\pi} \,,$$

so for $n > \frac{32M^2}{\epsilon^2 \pi}$, $\|h - h_n\|_2 < \frac{\epsilon}{2}$. It follows that for any such large n, $\|f - h_n\| < \epsilon$ and thus $g = h_n$ has the required properties.

6. Since $|f_n(x)| < M$ for all x, we also have

$$|f(x)| = |\lim_{n \to +\infty} f_n(x)| \le M$$

for all x. Hence

$$|f - f_n| \le |f| + |f_n| \le 2M$$

and the constant function $g = 2M$ is integrable on $[a, b]$. Thus we have that $h_n = |f - f_n| \to 0$, pointwise, as $n \to +\infty$ and $|h_n| \le g$, where g is integrable. Therefore the Dominated Convergence Theorem tells us that

$$\lim_{n \to +\infty} \int_a^b |f - f_n| \, d\mu = \lim_{n \to +\infty} \int_a^b h_n \, d\mu = \int_a^b 0 \, d\mu = 0.$$

7. Write $\phi(n) = (p(n), q(n))$. We observe that $q(n) \to +\infty$ as $n \to +\infty$.

(a) We have

$$\int_0^1 |g_n| \, d\mu = \int_0^1 |f_{\phi(n)}| \, d\mu = \int_{(p(n)-1)/q(n)}^{(p(n)+1)/q(n)} d\mu = \mu \left(\left[\frac{p(n) - 1}{q(n)}, \frac{p(n) + 1}{q(n)} \right] \right) = \frac{2}{q(n)} \to 0,$$

as $n \to +\infty$.

(b) Let $x \in [0, 1]$. Given $i \in \mathbb{N}$, we can choose $p(i, x) < i$ such that

$$x \in \left[\frac{p(i, x) - 1}{i}, \frac{p(i, x) + 1}{i} \right].$$

Define $n_i = \phi^{-1}(p(i, x), i)$. Then

$$g_{n_i}(x) = f_{(p(i,x),i)}(x) = 1,$$

for all $i \ge 1$.