For each $p > 0$ define $g_p : [-\pi, \pi] \to \mathbb{R}$ by

$$g_p(x) = \begin{cases} \frac{|x|}{p} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

A simple modification of the proof for $f_p$ from Example 4, Sheet 5, shows that $g_p$ is integrable if and only if $p < 1$. Thus the function $f = \frac{g_{1/2}}{2}$ is integrable. However, $|g_{1/2}|^2 = g_1$, which is not integrable. Thus, $f = \frac{g_{1/2}}{2}$ is not square integrable.

Note that if $\|f\|_2 \|g\|_2 = 0$ then $\|f\|_2 = 0$ or $\|g\|_2 = 0$ so $f = 0 \mu$-a.e. or $g = 0 \mu$-a.e. and we have equality both in (i) and (ii). Assume $\|f\|_2 \|g\|_2 \neq 0$.

(i) Let $f_0 = \frac{f}{\|f\|_2}$, $g_0 = \frac{g}{\|g\|_2}$. Noting that $\|f\|_2$ and $\|g\|_2$ are constants and applying the corollary\(^1\) to Lemma 4.1 to $f_0, g_0$, we get

$$\frac{2}{\pi} \int_{-\pi}^{\pi} \frac{|fg|}{\|f\|_2 \|g\|_2} d\mu \leq \left( \frac{\|f\|_2^2}{\|f\|_2^2} + \frac{\|g\|_2^2}{\|g\|_2^2} \right) = 2,$$

and rearranging,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |fg| d\mu \leq \|f\|_2 \|g\|_2,$$

with equality if and only if $|f| = (\|f\|_2 \|g\|_2) |g| \mu$-a.e.. If $|f| = c |g|$ \mu-a.e. for an arbitrary $c$ then $|f|^2 = c^2 |g|^2$ \mu-a.e. and so

$$\|f\|_2^2 = c^2 \|g\|_2^2.$$

\(^1\)Alternatively we can apply Lemma 4.1 rather than its corollary to functions $f_0 = \frac{f}{\|f\|_2}$, $g_0 = \frac{g}{\|g\|_2}$ obtaining again

$$\frac{2}{\pi} \int_{-\pi}^{\pi} \frac{|fg|}{\|f\|_2 \|g\|_2} d\mu = \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{|f| \|g|}{\|f\|_2 \|g\|_2} d\mu \leq \left( \frac{\|f\|_2^2}{\|f\|_2^2} + \frac{\|g|_2^2}{\|g\|_2^2} \right) = 2,$$

(since $\|f\|_2 = \|f\|_2$, $\|g\|_2 = \|g\|_2$) with equality if and only if $|f| = (\|f\|_2 \|g\|_2) |g| \mu$-a.e., and argue as above.
Thus we necessarily have $c = \|f\|_2/\|g\|_2$ and so the condition for equality becomes $|f| = c|g| \mu$-a.e., as required.

(ii) We have
\[
\left| \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, d\mu \right| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f| |g| \, d\mu \leq \|f\|_2 \|g\|_2,
\]
by Hölder’s inequality. However, this does not give us the correct condition for equality.

Applying Lemma 4.1 to functions $tf$ and $g$ with $t = \frac{1}{\|f\|_2} \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, d\mu$ we get
\[
\frac{2}{\pi^2} \frac{1}{\|f\|_2^2} \left( \int_{-\pi}^{\pi} f g \, d\mu \right)^2 \leq \left( \frac{1}{\|f\|_2^2} \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, d\mu \right)^2 \|f\|_2^2 + \|g\|_2^2,
\]
and rearranging gives
\[
\left| \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, d\mu \right| \leq \|f\|_2 \|g\|_2.
\]
Furthermore, we get equality if and only if $tf = g \mu$-a.e., that is,
\[
\left( \frac{1}{\|f\|_2^2} \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, d\mu \right) f = g \mu-a.e.
\]
and, because in this case
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f g \, d\mu = \pm \|f\|_2 \|g\|_2,
\]
this simplifies to
\[
f = \frac{\|f\|_2^2}{\|g\|_2^2} g \mu-a.e. \quad \text{or} \quad f = -\frac{\|f\|_2^2}{\|g\|_2^2} g \mu-a.e.
\]
As in part (i), we finish by noting that if $f = cg$ for an arbitrary $c \in \mathbb{R}$ then $|f|^2 = c^2 |g|^2$ and so $c = \pm \|f\|_2/\|g\|_2$. Thus the condition for equality becomes $f = cg \mu$-a.e., for some $c \in \mathbb{R}$, as required.

3. Suppose that $f : [-\pi, \pi] \to \mathbb{R}$ is square integrable. The constant function $g = 1$ is also square integrable. Therefore, by Hölder’s inequality,
\[
\int_{-\pi}^{\pi} |f| \, d\mu = \int_{-\pi}^{\pi} |f \cdot 1| \, d\mu \leq \pi \|f\|_2 \|1\|_2 = \pi \sqrt{2} \|f\|_2 < +\infty.
\]
Thus, $f$ is integrable.

4. Suppose that $f \in L^2([-\pi, \pi], \mu, \mathbb{R})$. If $g(x) = x^n$ then also have $g \in L^2([-\pi, \pi], \mu, \mathbb{R})$.
Applying Hölder’s inequality,
\[
\left( \int_{-\pi}^{\pi} |x^n f(x)| \, d\mu \right)^2 = \left( \int_{-\pi}^{\pi} |f g| \, d\mu \right)^2 \leq \pi^2 \|f\|_2^2 \|g\|_2^2 = \pi^2 \|f\|_2^2 \frac{1}{\pi} \int_{-\pi}^{\pi} x^{2n} \, d\mu
\]
\[
= \pi^2 \|f\|_2^2 \frac{1}{(2n+1)\pi} (\pi^{2n+1} - (-\pi)^{2n+1})
\]
\[
= \pi^2 \|f\|_2^2 \frac{2\pi^{2n+1}}{(2n+1)\pi} = \frac{2\pi^{2n+1}}{2n+1} \int_{-\pi}^{\pi} |f|^2 \, d\mu.
\]
5. By Theorem 4.5, we can find a continuous function \( h : [\pi, \pi] \to \mathbb{R} \) such that \( \|f - h\|_2 < \epsilon/2 \). Because \( h \) is continuous, it is bounded, i.e. there exists \( M > 0 \) such that \( |h(x)| \leq M \) for all \( x \in [\pi, \pi] \). For \( n \geq 1 \), define continuous functions \( h_n : [\pi, \pi] \to \mathbb{R} \) such that

1. for \( 0 \leq x \leq \pi + 1/n \), \( h_n(x) = 0 \) and \( h_n(\pi + 1/n) = h(\pi + 1/n) \);
2. for \( -\pi + 1/n < x < \pi - 1/n \), \( h_n(x) = h(x) \);
3. for \( \pi - 1/n \leq x \leq \pi \), \( h_n(x) = 0 \).

Clearly, \( h_n(\pi) = 0 \) and \( |h_n(x)| \leq M \) for all \( x \in [-\pi, \pi] \). By Minkowski Inequality, we have

\[
\|f - h_n\|_2 \leq \|f - h\|_2 + \|h - h_n\|_2.
\]

Since \( (h - h_n)^2 \) is 0 except on \( [\pi, \pi + 1/n] \cup [\pi - 1/n, \pi] \) where it is less or equal to \( 4M^2 \), we have

\[
\|h - h_n\|_2^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} (h(x) - h_n(x))^2 \, d\mu \leq \frac{4M^2}{\pi} \left( \left[ -\pi, \pi + \frac{1}{n} \right] \cup \left[ \pi - \frac{1}{n}, \pi \right] \right) = \frac{8M^2}{n\pi},
\]

so for \( n > \frac{32M^2}{\epsilon^2 \pi} \), \( \|h - h_n\|_2 < \frac{\epsilon}{2} \). It follows that for any such large \( n \), \( \|f - h_n\| < \epsilon \) and thus \( g = h_n \) has the required properties.

6. Since \( |f_n(x)| < M \) for all \( x \), we also have

\[
|f(x)| = \left| \lim_{n \to +\infty} f_n(x) \right| \leq M
\]

for all \( x \). Hence

\[
|f - f_n| \leq |f| + |f_n| \leq 2M
\]

and the constant function \( g = 2M \) is integrable on \([a, b]\). Thus we have that \( h_n = |f - f_n| \to 0 \), pointwise, as \( n \to +\infty \) and \( |h_n| \leq g \), where \( g \) is integrable. Therefore the Dominated Convergence Theorem tells us that

\[
\lim_{n \to +\infty} \int_a^b |f - f_n| \, d\mu = \lim_{n \to +\infty} \int_a^b h_n \, d\mu = \int_a^b 0 \, d\mu = 0.
\]

7. Write \( \phi(n) = (p(n), q(n)) \). We observe that \( q(n) \to +\infty \) as \( n \to +\infty \).

(a) We have

\[
\int_0^1 |g_n| \, d\mu = \int_0^1 |f_{\phi(n)}| \, d\mu = \int_0^1 (p(n) + 1/q(n)) \, d\mu = \mu \left( \left[ \frac{p(n) - 1}{q(n)}, \frac{p(n) + 1}{q(n)} \right] \right) = \frac{2}{q(n)} \to 0,
\]

as \( n \to +\infty \).

(b) Let \( x \in [0, 1] \). Given \( i \in \mathbb{N} \), we can choose \( p(i, x) < i \) such that

\[
x \in \left[ \frac{p(i, x) - 1}{i}, \frac{p(i, x) + 1}{i} \right].
\]

Define \( n_i = \phi^{-1}(p(i, x), i) \). Then

\[
g_{n_i}(x) = f_{(p(i, x), i)}(x) = 1,
\]

for all \( i \geq 1 \).