1. (i) Let \( f_n \) and \( g_n \) be increasing sequences of simple functions converging pointwise to \( f \) and \( g \), respectively. Then \( f_n + g_n \) are increasing and converge pointwise to \( f + g \). By the definition of the integral, basic properties of integrals of simple functions and basic properties of limits,

\[
\int (f + g) \, d\mu = \lim_{n \to +\infty} \int (f_n + g_n) \, d\mu = \lim_{n \to +\infty} \left( \int f_n \, d\mu + \int g_n \, d\mu \right) = \lim_{n \to +\infty} \int f_n \, d\mu + \lim_{n \to +\infty} \int g_n \, d\mu = \int f \, d\mu + \int g \, d\mu.
\]

(ii) Let \( f_n \) be an increasing sequence of simple functions converging pointwise to \( f \). Then \( cf_n \) is an increasing sequence of simple functions converging pointwise to \( cf \) and

\[
\int cf \, d\mu = \lim_{n \to +\infty} \int cf_n \, d\mu = \lim_{n \to +\infty} c \int f_n \, d\mu = c \lim_{n \to +\infty} \int f_n \, d\mu = c \int f \, d\mu.
\]

(iii) If \( f \geq g \) then \( f - g \) is a non-negative measurable function. We can choose an increasing sequence of non-negative simple functions \( h_n \) converging pointwise to \( f - g \). Then each \( \int h_n \, d\mu \geq 0 \) and so

\[
\int (f - g) \, d\mu = \lim_{n \to +\infty} \int h_n \, d\mu \geq 0.
\]

Then, by (i),

\[
\int f \, d\mu = \int (f - g) \, d\mu + \int g \, d\mu \geq \int g \, d\mu.
\]

2. If \( f, g \) are integrable then

\[
\int |f + g| \, d\mu \leq \int |f| \, d\mu + \int |g| \, d\mu < +\infty,
\]

so \( f + g \) is integrable.
If $f$ is integrable and $c \in \mathbb{R}$ then (using 1(ii))
\[
\int |cf| \, d\mu = |c| \int |f| \, d\mu < +\infty,
\]
so $cf$ is integrable.

(a) Write $f$ and $g$ in terms of their positive and negative parts: $f = f^+ - f^-$, $g = g^+ - g^-$. Now, $f + g = (f + g)^+ - (f + g)^-$ and $f + g = f^+ - f^- + g^+ - g^-$ so that
\[
(f + g)^+ + f^- + g^- = f^+ + g^+ + (f + g)^-.
\]
Hence,
\[
\int (f + g)^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu = \int f^+ \, d\mu + \int g^+ \, d\mu + \int (f + g)^- \, d\mu.
\]
Rearrangement gives the result.

(b) Again write $f = f^+ - f^-$. If $c \geq 0$ then $(cf)^+ = cf^+$, $(cf)^- = cf^-$. Then
\[
\int cf \, d\mu = \int cf^+ \, d\mu - \int cf^- \, d\mu = c \int f^+ \, d\mu - c \int f^- \, d\mu = c \int f^+ \, d\mu - c \int f^- \, d\mu = c \int f \, d\mu.
\]
If $c < 0$ then $(cf)^+ = -cf^-$, $(cf)^- = -cf^+$ and a similar argument gives the result.

3. The function $|f - g|$ is measurable, so $E$ is a measurable set. Thus
\[
h := 2M \chi_E + \epsilon \chi_{[0, 1] \setminus E}
\]
is a non-negative simple function with $|f - g| < h$. (We have used that $|f(x) - g(x)| \leq |f(x)| + |g(x)| < 2M$.) Thus (using Proposition 3.21),
\[
\int |f - g| \, d\mu < \int h \, d\mu = 2M \mu(E) + \epsilon \mu([0, 1] \setminus E) \leq 2M \mu(E) + \epsilon,
\]
as required.

4. Suppose $p < 1$. Modify $f_p$ so that $f_p(0) = 0$ – we know this does not affect the integral. We have $f_p \leq g$, where $g$ is the non-negative function
\[
\sum_{n=1}^{\infty} (n + 1)^p \chi_{[1/(n+1), 1/n]}.
\]
Then
\[
\int f_p \, d\mu \leq \int g \, d\mu.
\]
and, because \( g \) is the limit of the increasing sequence of simple functions

\[
g_N = \sum_{n=1}^{N} (n + 1)^p \chi_{(1/(n+1), 1/n]},
\]

we have

\[
\int g \, d\mu = \sum_{n=1}^{\infty} (n + 1)^p \mu \left( \left[ \frac{1}{n+1}, \frac{1}{n} \right] \right) = \sum_{n=1}^{\infty} (n + 1)^p \left( \frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{\infty} \frac{(n + 1)^p}{n(n + 1)} \leq \sum_{n=1}^{\infty} \frac{1}{n^{2-p}},
\]

which converges since \( p < 1 \implies 2 - p > 1 \). Thus \( f_p \) is integrable.

Now suppose \( p \geq 1 \). We have \( f_p \geq h \), where \( h \) is the non-negative function

\[
\sum_{n=1}^{\infty} n^p \chi_{(1/(n+1), 1/n]}.
\]

Then

\[
\int f_p \, d\mu \geq \int h \, d\mu
\]

and, because \( h \) is the limit of the increasing sequence of simple functions

\[
h_N = \sum_{n=1}^{N} n^p \chi_{(1/(n+1), 1/n]},
\]

we have

\[
\int h \, d\mu = \sum_{n=1}^{\infty} n^p \mu \left( \left[ \frac{1}{n+1}, \frac{1}{n} \right] \right) = \sum_{n=1}^{\infty} n^p \left( \frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{\infty} \frac{n^p}{n(n + 1)} \geq \sum_{n=1}^{\infty} \frac{1}{2n^{2-p}},
\]

which diverges since \( p \geq 1 \implies 2 - p \leq 1 \). Thus \( f_p \) is not integrable.

5. Easy one this! Let

\[
f(x) = \begin{cases} 
+\infty & \text{if } x \in \mathbb{Q} \cap [0, 1] \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( f(x) \) is infinite infinitely often but \( f = 0 \) \( \mu \)-a.e., so \( f \) is integrable.

6. As in the hint, consider the set

\[
E(n, m) = \bigcup_{k=n}^{\infty} \left\{ x \in [0, 1] : \| f_k(x) - f(x) \| \geq \frac{1}{m} \right\}
\]
and fix m. The sets have the property that

\[ E(n + 1, m) \subset E(n, m). \]

Pointwise convergence of the \( f_n \) means that, for each \( x \in [0, 1] \), \( \lim_{n \to +\infty} f_n(x) = f(x) \).

So each point \( x \) cannot lie in infinitely many of the sets \( E(n, m) \) (because \( |f_n(x) - f(x)| \) has to eventually be less than \( 1/m \)). Thus

\[
\bigcap_{n=1}^{\infty} E(n, m) = \emptyset.
\]

Since \( \mu(E(1, m)) \leq 1 \), we have (using 7(ii) on Sheet 3)

\[
0 = \mu \left( \bigcap_{n=1}^{\infty} E(n, m) \right) = \lim_{n \to +\infty} \mu(E(n, m)).
\]

Thus, in particular, given \( \epsilon > 0 \), we can find \( n_m \) such that

\[
\mu(E(n_m, m)) < \frac{\epsilon}{2^m}.
\]

Again following the hint, take

\[
A = \bigcup_{m=1}^{\infty} E(n_m, m).
\]

Then, by subadditivity,

\[
\mu(A) < \sum_{m=1}^{\infty} \frac{\epsilon}{2^m} = \epsilon.
\]

For all \( x \notin A \), we have that, for each \( m \geq 1 \) and for all \( k \geq n_m \)

\[
|f_k(x) - f(x)| < \frac{1}{m},
\]

i.e. \( f_k \) converges uniformly to \( f \) on \([0, 1] \setminus A\).