MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Solution Sheet 5

1.(i) Let f_n and g_n be increasing sequences of simple functions converging pointwise to f and g, respectively. Then $f_n + g_n$ are increasing and converge pointwise to $f+g$. By the definition of the integral, basic properties of integrals of *simple* functions and basic properties of limits,

$$\begin{aligned}\int (f+g)\,d\mu &= \lim_{n\to+\infty} \int (f_n+g_n)\,d\mu = \lim_{n\to+\infty} \left(\int f_n\,d\mu + \int g_n\,d\mu \right) \\ &= \lim_{n\to+\infty} \int f_n\,d\mu + \lim_{n\to+\infty} \int g_n\,d\mu = \int f\,d\mu + \int g\,d\mu.\end{aligned}$$

(ii) Let f_n be an increasing sequence of simple functions converging pointwise to f. Then cf_n is an increasing sequence of simple functions converging pointwise to cf and

$$\int cf\,d\mu = \lim_{n\to+\infty} \int cf_n\,d\mu = \lim_{n\to+\infty} c \int f_n\,d\mu = c \lim_{n\to+\infty} \int f_n\,d\mu = c \int f\,d\mu.$$

(iii) If $f \geq g$ then $f-g$ is a non-negative measurable function. We can choose an increasing sequence of non-negative simple functions h_n converging pointwise to $f-g$. Then each $\int h_n\,d\mu \geq 0$ and so

$$\int f - g\,d\mu = \lim_{n\to+\infty} \int h_n\,d\mu \geq 0.$$

Then, by (i),

$$\int f\,d\mu = \int (f-g)\,d\mu + \int g\,d\mu \geq \int g\,d\mu.$$

2. If f, g are integrable then

$$\int |f+g|\,d\mu \leq \int |f|\,d\mu + \int |g|\,d\mu < +\infty,$$

so $f+g$ is integrable.

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If f is integrable and $c \in \mathbb{R}$ then (using 1(ii))

$$\int |cf|\, d\mu = |c| \int |f|\, d\mu < +\infty,$$

so cf is integrable.
(a) Write f and g in terms of their positive and negative parts: $f = f^+ - f^-$, $g = g^+ - g^-$. Now, $f + g = (f+g)^+ - (f+g)^-$ and $f + g = f^+ - f^- + g^+ - g^-$ so that

$$(f+g)^+ + f^- + g^- = f^+ + g^+ + (f+g)^-.$$

Hence,

$$\int (f+g)^+\, d\mu + \int f^-\, d\mu + \int g^-\, d\mu = \int f^+\, d\mu + \int g^+\, d\mu + \int (f+g)^-\, d\mu.$$

Rearrangement gives the result.
(b) Again write $f = f^+ - f^-$. If $c \geq 0$ then $(cf)^+ = cf^+$, $(cf)^- = cf^-$. Then

$$\begin{aligned} \int cf\, d\mu &= \int cf^+\, \mu - \int cf^-\, d\mu = c\int f^+\, d\mu - c\int f^-\, d\mu \\ &= c\left(\int f^+\, d\mu - \int f^-\, d\mu\right) = c\int f\, d\mu. \end{aligned}$$

If $c < 0$ then $(cf)^+ = -cf^-$, $(cf)^- = -cf^+$ and a similar argument gives the result.

3. The function $|f - g|$ is measurable, so E is a measurable set. Thus

$$h := 2M\chi_E + \epsilon\chi_{[0,1]\backslash E}$$

is a non-negative simple function with $|f - g| < h$. (We have used that $|f(x) - g(x)| \leq |f(x)| + |g(x)| < 2M$.) Thus (using Proposition 3.21),

$$\int |f - g|\, d\mu < \int h\, d\mu = 2M\mu(E) + \epsilon\mu([0,1]\backslash E) \leq 2M\mu(E) + \epsilon,$$

as required.

4. Suppose $p < 1$. Modify f_p so that $f_p(0) = 0$ – we know this does not affect the integral. We have $f_p \leq g$, where g is the non-negative function

$$\sum_{n=1}^{\infty} (n+1)^p \chi_{[1/(n+1), 1/n]}.$$

Then

$$\int f_p\, d\mu \leq \int g\, d\mu$$

and, because g is the limit of the increasing sequence of simple functions

$$g_N = \sum_{n=1}^{N} (n+1)^p \chi_{[1/(n+1),1/n]},$$

we have

$$\begin{aligned}\int g\, d\mu &= \sum_{n=1}^{\infty} (n+1)^p \mu\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right) = \sum_{n=1}^{\infty} (n+1)^p \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \sum_{n=1}^{\infty} \frac{(n+1)^p}{n(n+1)} \le \sum_{n=1}^{\infty} \frac{1}{n^{2-p}},\end{aligned}$$

which converges since $p < 1 \implies 2 - p > 1$. Thus f_p is integrable.

Now suppose $p \ge 1$. We have $f_p \ge h$, where h is the non-negative function

$$\sum_{n=1}^{\infty} n^p \chi_{[1/(n+1),1/n]}.$$

Then

$$\int f_p\, d\mu \ge \int h\, d\mu$$

and, because h is the limit of the increasing sequence of simple functions

$$h_N = \sum_{n=1}^{N} n^p \chi_{[1/(n+1),1/n]},$$

we have

$$\begin{aligned}\int h\, d\mu &= \sum_{n=1}^{\infty} n^p \mu\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right) = \sum_{n=1}^{\infty} n^p \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \sum_{n=1}^{\infty} \frac{n^p}{n(n+1)} \ge \sum_{n=1}^{\infty} \frac{1}{2n^{2-p}},\end{aligned}$$

which diverges since $p \ge 1 \implies 2 - p \le 1$. Thus f_p is not integrable.

5. Easy one this! Let

$$f(x) = \begin{cases} +\infty \text{ if } x \in \mathbb{Q} \cap [0,1] \\ 0 \text{ otherwise.} \end{cases}$$

Then $f(x)$ is infinite infinitely often but $f = 0$ μ-a.e., so f is integrable.

6. As in the hint, consider the set

$$E(n,m) = \bigcup_{k=n}^{\infty} \left\{ x \in [0,1] : |f_k(x) - f(x)| \ge \frac{1}{m} \right\}$$

and fix m. The sets have the property that

$$E(n+1,m) \subset E(n,m).$$

Pointwise convergence of the f_n means that, for each $x \in [0,1]$, $\lim_{n\to+\infty} f_n(x) = f(x)$. So each point x cannot lie in infinitely many of the sets $E(n,m)$ (because $|f_n(x) - f(x)|$ has to eventually be less than $1/m$). Thus

$$\bigcap_{n=1}^{\infty} E(n,m) = \varnothing.$$

Since $\mu(E(1,m)) \leq 1$, we have (using 7(ii) on Sheet 3)

$$0 = \mu\left(\bigcap_{n=1}^{\infty} E(n,m)\right) = \lim_{n\to+\infty} \mu(E(n,m)).$$

Thus, in particular, given $\epsilon > 0$, we can find n_m such that

$$\mu(E(n_m,m)) < \frac{\epsilon}{2^m}.$$

Again following the hint, take

$$A = \bigcup_{m=1}^{\infty} E(n_m,m).$$

Then, by subadditivity,

$$\mu(A) < \sum_{m=1}^{\infty} \frac{\epsilon}{2^m} = \epsilon.$$

For all $x \notin A$, we have that, for each $m \geq 1$ and for all $k \geq n_m$

$$|f_k(x) - f(x)| < \frac{1}{m},$$

i.e. f_k converges uniformly to f on $[0,1]\backslash A$.