MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Solution Sheet 6

1. Clearly, each f_n is non-negative. Also, the sequence f_n is increasing. If $f(x) = +\infty$ then $f_n(x) = n$, for all $n \geq |x|$, so $\lim_{n\to+\infty} f_n(x) = +\infty$. If $f(x) < +\infty$ then $f_n(x) = f(x)$, for all $n \geq |x|$, $n \geq [f(x)] + 1$ ($[\cdot]$ = integer part), so $\lim_{n\to+\infty} f_n(x) = f(x)$. Thus, f_n converges to f pointwise, as $n \to +\infty$. Applying the Monotone Convergence Theorem gives

$$\lim_{n\to+\infty} \int_{\mathbb{R}} f_n \, d\mu = \int_{\mathbb{R}} f \, d\mu.$$

2. Write $g_n = \sum_{i=1}^{n} f_i$. Then g_n is an increasing sequence of non-negative measurable functions whose limit is $\sum_{n=1}^{\infty} f_n$. So, by the Monotone Convergence Theorem,

$$\lim_{n\to+\infty} \int g_n \, d\mu = \int \left(\sum_{n=1}^{\infty} f_n \right) d\mu \, .$$

But by additivity of the integral, $\int g_n \, d\mu = \sum_{i=1}^{n} \int f_i \, d\mu$. Therefore,

$$\sum_{n=1}^{\infty} \left(\int f_n \, d\mu \right) = \int \left(\sum_{n=1}^{\infty} f_n \right) d\mu.$$

3. $f_n(0) = 0$ and for $x \in (0,1]$ we have $0 < \frac{n\sqrt{x}}{1+n^2x^2} \leq \frac{n}{n^2x^2} = \frac{1}{nx^2} \to 0$ so $f_n \to 0$ as $n \to \infty$.

Also, for $x \in (0,1]$

$$\frac{n\sqrt{x}}{1+n^2x^2} \leq \frac{1}{2\sqrt{x}} \iff 2nx \leq 1 + n^2x^2 \iff 0 \leq (1-nx)^2$$

which shows that $f_n \leq G$. f_n are continuous and hence measurable and G is integrable (see the previous example sheet) so the result follows by the Dominated Convergence theorem.

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4. First we shall use the MCT. Write

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k^2}.$$

Clearly f_n, $n \geq 1$ is an increasing sequence of functions. They are non-negative and measurable (since they are continuous). Also,

$$\int f_n \, d\mu = \sum_{k=1}^{n} \int \frac{x^k}{k^2} \, d\mu = \sum_{k=1}^{n} \frac{1}{(k+1)k^2}$$

(the integrals are the same as Riemann integrals). By the MCT,

$$\int f \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu = \sum_{k=1}^{\infty} \frac{1}{(k+1)k^2},$$

and f is integrable because the infinite sum $\sum_{k=1}^{\infty} \frac{1}{(k+1)k^2}$ is finite.

Now we'll give a different solution using the DCT. Again write

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k^2}.$$

Clearly, $f(x) = \lim_{n \to +\infty} f_n(x)$ for all $x \in [0, 1]$. As above, the f_n are measurable. Furthermore, since for $x \in [0, 1]$,

$$\left| \frac{x^k}{k^2} \right| \leq \frac{1}{k^2},$$

we have

$$|f_n(x)| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}$$

and, since the infinite sum is finite, the RHS is a constant, integrable function. Then, by the DCT, f is integrable and

$$\int f \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu = \sum_{k=1}^{\infty} \frac{1}{(k+1)k^2}.$$