1. Clearly, each $f_n$ is non-negative. Also, the sequence $f_n$ is increasing. If $f(x) = +\infty$ then $f_n(x) = n$, for all $n \geq |x|$, so $\lim_{n \to +\infty} f_n(x) = +\infty$. If $f(x) < +\infty$ then $f_n(x) = f(x)$, for all $n \geq |x|$, $n \geq \lceil f(x) \rceil + 1$ ($\lceil \cdot \rceil = \text{integer part}$), so $\lim_{n \to +\infty} f_n(x) = f(x)$. Thus, $f_n$ converges to $f$ pointwise, as $n \to +\infty$. Applying the Monotone Convergence Theorem gives

$$\lim_{n \to +\infty} \int f_n \, d\mu = \int f \, d\mu.$$ 

2. Write $g_n = \sum_{i=1}^{n} f_i$. Then $g_n$ is an increasing sequence of non-negative measurable functions whose limit is $\sum_{n=1}^{\infty} f_n$. So, by the Monotone Convergence Theorem,

$$\lim_{n \to +\infty} \int g_n \, d\mu = \int \left( \sum_{n=1}^{\infty} f_n \right) \, d\mu.$$ 

But by additivity of the integral, $\int g_n \, d\mu = \sum_{i=1}^{n} \int f_i \, d\mu$. Therefore,

$$\sum_{n=1}^{\infty} \left( \int f_n \, d\mu \right) = \int \left( \sum_{n=1}^{\infty} f_n \right) \, d\mu.$$ 

3. $f_n(0) = 0$ and for $x \in (0, 1]$ we have $0 < \frac{n\sqrt{x}}{1 + n^2 x^2} \leq \frac{n}{n^2 x^2} = \frac{1}{n x^2} \to 0$ so $f_n \to 0$ as $n \to \infty$.

Also, for $x \in (0, 1]$

$$\frac{n\sqrt{x}}{1 + n^2 x^2} \leq \frac{1}{2\sqrt{x}} \iff 2nx \leq 1 + n^2 x^2 \iff 0 \leq (1 - nx)^2$$

which shows that $f_n \leq G$. $f_n$ are continuous and hence measurable and $G$ is integrable (see the previous example sheet) so the result follows by the Dominated Convergence theorem.
4. First we shall use the MCT. Write

\[ f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k^2}. \]

Clearly \( f_n, \ n \geq 1 \) is an increasing sequence of functions. They are non-negative and measurable (since they are continuous). Also,

\[
\int f_n \, d\mu = \sum_{k=1}^{n} \int \frac{x^k}{k^2} \, d\mu = \sum_{k=1}^{n} \frac{1}{(k+1)k^2}
\]

(the integrals are the same as Riemann integrals). By the MCT,

\[
\int f \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu = \sum_{k=1}^{\infty} \frac{1}{(k+1)k^2},
\]

and \( f \) is integrable because the infinite sum \( \sum_{k=1}^{\infty} \frac{1}{(k+1)k^2} \) is finite.

Now we’ll give a different solution using the DCT. Again write

\[ f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k^2}. \]

Clearly, \( f(x) = \lim_{n \to +\infty} f_n(x) \) for all \( x \in [0,1] \). As above, the \( f_n \) are measurable. Furthermore, since for \( x \in [0,1] \),

\[
\left| \frac{x^k}{k^2} \right| \leq \frac{1}{k^2},
\]

we have

\[
|f_n(x)| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}
\]

and, since the infinite sum is finite, the RHS is a constant, integrable function. Then, by the DCT, \( f \) is integrable and

\[
\int f \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu = \sum_{k=1}^{\infty} \frac{1}{(k+1)k^2}.
\]