MATH31011/MATH41011/MATH61011: FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Solution Sheet 4

1. Since f and g are simple functions, we can find two collections of pairwise disjoint sets $A_1, \dots, A_n$ and $B_1, \dots, B_n$, and two collections of real numbers $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$, such that $\bigcup_{i=1}^{n} A_i = [0,1]$, $\bigcup_{j=1}^{m} B_j = [0,1]$,

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \quad \text{and} \quad g = \sum_{j=1}^{m} \beta_j \chi_{B_j}.$$

(a) We have

$$kf = \sum_{i=1}^{n} (k\alpha_i) \chi_{A_i},$$

so it is easy to see that kf is simple.

(b) We have

$$|f| = \sum_{i=1}^{n} |\alpha_i| \chi_{A_i},$$

so again it is easy to see that $|f|$ is simple.

(c) The intersections $A_i \cap B_j$, $i = 1, \dots, n$, $j = 1, \dots, m$ form a collection of pairwise disjoint sets with $\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \cap B_j) = [0,1]$. We have

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) \chi_{A_i \cap B_j}.$$

Thus $f + g$ is simple.

(d) Similarly to (c), we can write

$$fg = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \beta_j) \chi_{A_i \cap B_j}.$$

Thus fg is simple.

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2. As in question 1, write

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \quad \text{and} \quad g = \sum_{j=1}^{m} \beta_j \chi_{B_j}.$$

(i) Adapting 1(c), we have

$$af + bg = \sum_{i=1}^{n} \sum_{j=1}^{m} (a\alpha_i + b\beta_j) \chi_{A_i \cap B_j}.$$

Note that, for each i, j,

$$\bigcup_{j=1}^{m} A_i \cap B_j = A_i \quad \text{and} \quad \bigcup_{i=1}^{n} A_i \cap B_j = B_j$$

and that these unions are of disjoint sets. By definition,

$$\begin{aligned}
\int (af + bg)\, d\mu &= \sum_{i=1}^{n} \sum_{j=1}^{m} (a\alpha_i + b\beta_j) \mu(A_i \cap B_j) \\
&= \sum_{i=1}^{n} a\alpha_i \sum_{j=1}^{m} \mu(A_i \cap B_j) + \sum_{j=1}^{m} b\beta_j \sum_{i=1}^{n} \mu(A_i \cap B_j) \\
&= a \sum_{i=1}^{n} \alpha_i \mu \left(\bigcup_{j=1}^{m} A_i \cap B_j \right) + b \sum_{j=1}^{m} \beta_j \mu \left(\bigcup_{i=1}^{n} A_i \cap B_j \right) \\
&= a \sum_{i=1}^{n} \alpha_i \mu(A_i) + b \sum_{j=1}^{n} \beta_j \mu(B_j) \\
&= a \int f\, d\mu + b \int g\, d\mu,
\end{aligned}$$

as required.

(ii) If $f(x) \leq g(x)$ for all x then, whenever $A_i \cap B_j \neq \emptyset$, $\alpha_i \leq \beta_j$. Thus,

$$\begin{aligned}
\int f\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i) &= \sum_{i=1}^{n} \alpha_i \mu\left(\bigcup_{j=1}^{m} A_i \cap B_j\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \mu(A_i \cap B_j) \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j \mu(A_i \cap B_j) \\
&= \sum_{j=1}^{m} \beta_j \mu\left(\bigcup_{i=1}^{n} A_i \cap B_j\right) \\
&= \sum_{j=1}^{m} \beta_j \mu(B_j) \\
&= \int g\, d\mu,
\end{aligned}$$

as required.

(iii) We have

$$\left|\int f\, d\mu\right| = \left|\sum_{i=1}^{n} \alpha_i \mu(A_i)\right| \leq \sum_{i=1}^{n} |\alpha_i| \mu(A_i) = \int |f|\, d\mu,$$

as required.

3. Since the E_i might not be disjoint, we cannot use the definition of the integral of a simple function. However, we have $f = f_1 + \cdots + f_n$, where f_i is the simple function

$$f_i = \alpha_i \chi_{E_i} + (0 \times \chi_{[0,1]\setminus E_i}).$$

By question 2,

$$\int f\, d\mu = \sum_{i=1}^{n} \int f_i\, d\mu = \sum_{i=1}^{n} (\alpha_i \mu(E_i) + (0 \times \mu([0,1]\setminus E_i))) = \sum_{i=1}^{n} \alpha_i \mu(E_i),$$

as required.

4. Suppose that $f(x) = c$ for all $x \in [0,1]$. Then, for $a \in \mathbb{R}$,

$$\{x \in [0,1] : f(x) \leq a\} = \begin{cases} \emptyset & \text{if } a < c \\ [0,1] & \text{if } a \geq c. \end{cases}$$

Both $\emptyset$ and $[0,1]$ are in $\mathcal{M}([0,1])$, so f is measurable.

5.(i) Suppose that f is simple and write $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, where the measurable sets E_i are disjoint and assume that

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n.$$

Then, for $a \in \mathbb{R}$,

$$\{x \in [0,1] : f(x) \le a\} = \begin{cases} \emptyset & \text{if } a < \alpha_1 \\ \bigcup_{i=1}^{m} E_i & \text{if } \alpha_m \le a < \alpha_{m+1} \\ [0,1] & \text{if } \alpha_n \le a. \end{cases}$$

All the sets on the Right Hand Side are in $\mathcal{M}([0,1])$, so f is measurable.

(ii) Suppose that $f : [0,1] \to \mathbb{R}^*$ is continuous. Note that it means that f takes only finite values. Define

$$\bar{f}(x) = \begin{cases} f(0) & \text{if } x < 0 \\ f(x) & \text{if } x \in [0,1] \\ f(1) & \text{if } x > 1\,. \end{cases}$$

Since $\bar{f}$ takes only finite values, we have $\bar{f}^{-1}([-\infty, a)) = \bar{f}^{-1}((-\infty, a))$. Now, $(-\infty, a)$ is an open set in $\mathbb{R}$, so, since $\bar{f} : \mathbb{R} \to \mathbb{R}$ is continuous, $\bar{f}^{-1}((-\infty, a))$ is an open set and is hence in $\mathcal{M}$. Consequently, $f^{-1}((-\infty, a)) = \bar{f}^{-1}((-\infty, a)) \cap ([0,1]) \in \mathcal{M}([0,1])$ and hence f is measurable.

6. Suppose that f is increasing. Let $a \in \mathbb{R}$ and consider

$$f^{-1}((a, \infty]) = \{x \in [0,1] : f(x) > a\}\,.$$

If $x \in f^{-1}((a, \infty])$ and $b \ge x$ then since $f(b) \ge f(x) > a$ we have also $b \in f^{-1}((a, \infty])$. Thus we see that $f^{-1}((a, \infty])$ is equal either to $\emptyset$ or to $(b_a, 1]$ or $[b_a, 1]$, where $b_a = \inf f^{-1}((a, \infty])$. In each case, this is in $\mathcal{M}([0,1])$, so we conclude from Lemma 3.10 that f is measurable.

7. For $a \in \mathbb{R}$ we have

$$\{x \in [0,1] : f(x) = a\} = \{x \in [0,1] : f(x) \le a\} \cap \{x \in [0,1] : f(x) \ge a\}.$$

Since both of the sets on the Right Hand Side are in $\mathcal{M}([0,1])$, so is their intersection. Hence $\{x \in [0,1] : f(x) = a\}$ is a measurable set.

Also,

$$\{x \in [0,1] : f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \{x \in [0,1] : f(x) \ge n\}$$

and since all the sets $\{x \in [0,1] : f(x) \ge n\}$ are in the σ-algebra $\mathcal{M}([0,1])$, so is their intersection. Similarly for

$$\{x \in [0,1] : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in [0,1] : f(x) \le -n\}\,.$$

8. For any $c \in \mathbb{R}$,

$$\{x \in [0,1] : f_a(x) \le c\} = \begin{cases} \mathbb{R} & \text{if } c \ge a \\ \{x \in [0,1] : f(x) \le c\} & \text{if } c < a \end{cases}.$$

Since f is measurable, both sets on the Right Hand Side belong to $\mathcal{M}([0,1])$, so f_a is measurable by Lemma 3.10.

9. Let

$$f = \sum_{i=1}^{k} \alpha_i \chi_{E_i},$$

where $E_1, \ldots, E_k$ are disjoint measurable sets with union $[0,1]$. Since each $E_i \subset [0,1]$, we have $\mu(E_i) < +\infty$. By question 8 on Sheet 3, we can find an open set $U_i \supset E_i$ such that $\mu(U_i \backslash E_i) < \epsilon/2k$. By question 6 on Sheet 3, U_i is a countable union of disjoint open intervals: $U_i = \bigcup_{j=1}^{\infty} I_{i,j}$. Now, by countable additivity,

$$\sum_{j=1}^{\infty} \mu(I_{i,j}) = \mu(U_i) = \mu(E_i) + \mu(U_i \backslash E_i) < \mu(E_i) + \epsilon/2k < +\infty,$$

i.e. $\sum_{j=1}^{\infty} \mu(I_{i,j})$ converges. Thus, we can choose N_i such that

$$\sum_{j=N_i+1}^{\infty} \mu(I_{i,j}) < \frac{\epsilon}{2k}.$$

Then

$$g_i = \sum_{j=1}^{N_i} \alpha_i \chi_{I_{i,j}}$$

is a step function and so

$$g = \sum_{i=1}^{k} g_i$$

is a step function. Furthermore, $g(x) = f(x)$ except possibly for

$$x \in A := \left(\bigcup_{i=1}^{k} \bigcup_{j=N_i+1}^{\infty} I_{i,j} \right) \cup \left(\bigcup_{i=1}^{k} U_i \backslash E_i \right).$$

Now

$$\mu\left(\bigcup_{i=1}^{k} \bigcup_{j=N_i+1}^{\infty} I_{i,j} \right) \le \sum_{i=1}^{k} \sum_{j=N_i+1}^{\infty} \mu(I_{i,j}) < k\frac{\epsilon}{2k} = \frac{\epsilon}{2}$$

and

$$\mu\left(\bigcup_{i=1}^{k} U_i \backslash E_i \right) \le \sum_{i=1}^{k} \mu(U_i \backslash E_i) < k\frac{\epsilon}{2k} = \frac{\epsilon}{2}.$$

Thus $\mu(A) < \epsilon$, as required.