1. Let $E \subset \mathbb{R}$ be a countable set. Assume $E$ is countably infinite – the finite case is even easier. Then we can write $E = \{x_n\}_{n=1}^\infty$. Choose $\epsilon > 0$. Then

$$\left\{ \left( x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}} \right) \right\}_{n=1}^\infty$$

is a countable collection of open intervals such that

$$E \subset \bigcup_{n=1}^\infty \left( x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}} \right)$$

and

$$\sum_{n=1}^\infty l \left( x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}} \right) = \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon.$$

Thus, $E$ is a null set.

Note To make things fit the definition of a null set exactly, we could use intervals around $x_n$ of length $\frac{1}{2^{n+1}}$ to make the sum of the lengths $\frac{\epsilon}{2}$ and thus strictly less than $\epsilon$ as required by the definition in the Notes. However, the above clearly suffices.

2. We start with the unit interval 

$$F_0 = [0, 1].$$

Now define a new set

$$F_1 = F_0 \setminus (2/5, 3/5) = [0, 2/5] \cup [3/5, 1],$$

i.e., we obtain $F_1$ by deleting the open middle fifth of $F_0$. Next we obtain a new set $F_2$ by deleting the open middle fifths of each of the intervals making up $F_1$,

$$F_2 = \left[ 0, \frac{4}{25} \right] \cup \left[ \frac{6}{25}, \frac{2}{5} \right] \cup \left[ \frac{3}{5}, \frac{19}{25} \right] \cup \left[ \frac{21}{25}, 1 \right].$$
Continue in this way to obtain sets $F_n$, $n \geq 0$, where $F_n$ consists of $2^n$ disjoint closed intervals of length $(2/5)^n$, formed by deleting the middle thirds of the intervals making up $F_{n-1}$. The Middle Fifth Cantor set is defined to be the intersection of these sets:

$$F = \bigcap_{n=1}^{\infty} F_n.$$ 

Let $\{F_n^1, \ldots, F_n^{2^n}\}$ denote the $2^n$ intervals making up $F_n$. Enlarge these to open intervals $U_n^1, \ldots, U_n^{2^n}$ by setting

$$U_n^i = \left( a - \frac{\delta^n}{2}, b + \frac{\delta^n}{2} \right),$$

where $F_n^i = [a, b]$. Then

$$l(U_n^i) = \left( \frac{2}{5} \right)^n + \delta^n < \left( \frac{2}{5} + \delta \right)^n.$$ 

Now choose $\delta > 0$ sufficiently small that

$$\frac{2}{5} + \delta < \frac{1}{2},$$

i.e. take $\delta < 1/2 - 2/5 = 1/10 - \delta - 1/11$ will do.

Given $\epsilon > 0$, choose $n$ sufficiently large that

$$2^n \left( \frac{2}{5} + \delta \right)^n < \epsilon.$$ 

Then

$$F \subset \bigcup_{i=1}^{2^n} U_n^i$$

and

$$\sum_{i=1}^{2^n} l(U_n^i) < \sum_{i=1}^{2^n} \left( \frac{2}{5} + \delta \right)^n = 2^n \left( \frac{2}{5} + \delta \right)^n < \epsilon.$$ 

Thus, $F$ is a null set.

3. (a) Clearly $\emptyset \in \mathcal{A}$. If $A \in \mathcal{A}$ then either $A$ is countable or $[0,1] \setminus A$ is countable. So $[0,1] \setminus A$ is countable or $[0,1] \setminus ([0,1] \setminus A) = A$ is countable. Thus $[0,1] \setminus A \in \mathcal{A}$. Finally, suppose $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$. There are two possibilities:

(i) all the sets $A_j$ are countable. Then $\bigcup_{j=1}^{\infty} A_j$ is countable and so $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. 

(ii) at least one of the sets $A_j$ is uncountable. Call one of the uncountable ones $A^*$. Since $A^* \in \mathcal{A}$, we have that $[0,1] \setminus A^*$ is countable. Then, since

$$[0,1] \setminus \left( \bigcup_{j=1}^{\infty} A_j \right) \subset [0,1] \setminus A^*,$$

we see that $[0,1] \setminus \left( \bigcup_{j=1}^{\infty} A_j \right)$ is countable and so $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. 

2
(b) Clearly $\emptyset \in \mathcal{A}$. If $A \in \mathcal{A}$ then either $A$ is a null set or $[0, 1] \setminus A$ is a null set. So $[0, 1] \setminus A$ is a null set or $[0, 1] \setminus ( [0, 1] \setminus A) = A$ is a null set. Thus $[0, 1] \setminus A \in \mathcal{A}$. Finally, suppose \( \{A_j\}_{j=1}^\infty \subset \mathcal{A} \). There are two possibilities:

(i) all the sets $A_j$ are null sets. Choose $\epsilon > 0$. Then, for each $j$, we can find a countable collection of open intervals $\{I^j_n\}$ such that

$$A_j \subset \bigcup_{n} I^j_n \quad \text{and} \quad \sum_{n} l(I^j_n) < \frac{\epsilon}{2^j}.$$ 

We have

$$\bigcup_{j=1}^\infty A_j \subset \bigcup_{j=1}^\infty \bigcup_{n} I^j_n$$

and

$$\sum_{j=1}^\infty \sum_{n} l(I^j_n) < \sum_{j=1}^\infty \frac{\epsilon}{2^j} = \epsilon.$$ 

Thus $\bigcup_{j=1}^\infty A_j$ is a null set and so $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$.

(ii) at least one of the sets $A_j$ is not a null set. Call one of the non-null ones $A^*$. Since $A^* \in \mathcal{A}$, we have that $[0, 1] \setminus A^*$ is a null set. Then, since

$$[0, 1] \setminus \left( \bigcup_{j=1}^\infty A_j \right) \subset [0, 1] \setminus A^*,$$

we see that $[0, 1] \setminus \left( \bigcup_{j=1}^\infty A_j \right)$ is a null set and so $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$.

4. We have $\emptyset \in \mathcal{A}_\alpha$ for each $\alpha$, so $\emptyset \in \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$. Suppose that $E \in \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$. Then $E \in \mathcal{A}_\alpha$, for each $\alpha$, and so $X \setminus E \in \mathcal{A}_\alpha$, for each $\alpha$. Thus, $X \setminus E \in \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$. Finally, suppose that $\{E_j\}_{j=1}^\infty \subset \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$. Then $\{E_j\}_{j=1}^\infty \subset \mathcal{A}_\alpha$, for each $\alpha$, and so $\bigcup_{j=1}^\infty E_j \in \mathcal{A}_\alpha$, for each $\alpha$. Thus, $\bigcup_{j=1}^\infty E_j \in \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$.

5. Clearly $\emptyset \in \mathcal{A}(X)$. Suppose that $E \in \mathcal{A}(X)$. Then $E \in \mathcal{A}$ and $E \subset X$, so $X \setminus E = X \cap \mathbb{R} \setminus E \in \mathcal{A}$. Since $X \setminus E \subset X$, we have $X \setminus E \in \mathcal{A}(X)$. Finally, suppose that $\{E_j\}_{j=1}^\infty \subset \mathcal{A}(X)$. Then $\{E_j\}_{j=1}^\infty \subset \mathcal{A}$, so $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$. Furthermore, each $E_j \subset X$, so $\bigcup_{j=1}^\infty E_j \subset X$. Thus, $\bigcup_{j=1}^\infty E_j \in \mathcal{A}(X)$.

6. First, an argument that shows an open set is a countable union of open intervals (but doesn't show disjointness.) Let $U \subset \mathbb{R}$ be an open set. Consider the set of open intervals $(a, b)$ with $a, b \in \mathbb{Q}$. This is a countable collection since $\mathbb{Q} \times \mathbb{Q}$ is countable. Since $U$ is open, if $x \in U$ then $(x - \delta, x + \delta) \subset U$, for some $\delta > 0$. We may then choose rationals $a, b$ with $x - \delta < a < x < b < x + \delta$, so that $x \in (a, b) \subset U$. Let $\{(a_n, b_n)\}_{n=1}^\infty$ denote the countable set of open intervals which arise in this way. By construction,

$$U = \bigcup_{n=1}^\infty (a_n, b_n),$$
a countable union.

Now a solution to the whole question. Let $U \subset \mathbb{R}$ be an open set. For each $x \in U$, define an open interval $O(x)$ containing $x$ by

$$O(x) = (x - \delta_1(x), x + \delta_2(x)),$$

where

$$\delta_1(x) = \sup \{ \delta : (x - \delta, x] \subset U \},$$
$$\delta_2(x) = \sup \{ \delta : [x, x + \delta) \subset U \}.$$

Clearly,

$$U \subset \bigcup_{x \in U} O(x).$$

Now suppose that $O(x)$ is not a subset of $U$. Then there exists $z \notin U$ such that

$$x - \delta_1(x) < z < x \quad \text{or} \quad x < z < x + \delta_2(x).$$

This contradicts the definition of $\delta_1(x)$ or $\delta_2(x)$. Thus, $O(x) \subset U$ and so

$$\bigcup_{x \in U} O(x) \subset U.$$

Hence

$$\bigcup_{x \in U} O(x) = U.$$

If $x, y \in U$ and $O(x) \cap O(y) \neq \emptyset$ but $O(x) \neq O(y)$ then $O(x) \cup O(y)$ is an open interval containing $x$, contained in $U$ and strictly larger than $O(x)$. This contradicts the definition of $\delta_1(x)$ or $\delta_2(x)$. Thus, if $O(x) \cap O(y) \neq \emptyset$ then $O(x) = O(y)$. This shows that

$$\bigcup_{x \in U} O(x) = U$$

is a union of disjoint open intervals.

Since each $O(x)$ is an open interval, it contains a rational number $r \in \mathbb{Q}$. By the argument about, $O(r) = O(x)$. Thus, because $\mathbb{Q}$ is countable,

$$U = \bigcup_{r \in U \cap \mathbb{Q}} O(r)$$

is a countable union of disjoint open intervals.
7. (i) Let \( B_1 = A_1 \) and \( B_n = A_n \setminus A_{n-1} \) for \( n > 1 \). Then \( \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \), \( \bigcup_{i=1}^{\infty} B_i = A_n \) and \( \{B_n\}_{n=1}^{\infty} \) is a sequence of disjoint sets. Then

\[
\mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \mu \left( \bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n \to +\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n \to +\infty} \mu \left( \bigcup_{i=1}^{n} B_i \right) = \lim_{n \to +\infty} \mu(A_n).
\]

(ii) Since \( \{A_n\}_{n=1}^{\infty} \) is decreasing, the sequence \( A'_n = A_1 \setminus A_n \) is increasing, so by (i),

\[
\mu \left( \bigcup_{n=1}^{\infty} A'_n \right) = \lim_{n \to +\infty} \mu(A'_n) = \lim_{n \to +\infty} (\mu(A_1) - \mu(A_n)) \quad \text{(since } \mu(A_1) \text{ is finite)}.
\]

But

\[
\mu \left( \bigcup_{n=1}^{\infty} A'_n \right) = \mu \left( \bigcup_{n=1}^{\infty} (A_1 \setminus A_n) \right) = \mu \left( A_1 \setminus \bigcap_{n=1}^{\infty} A_n \right) = \mu(A_1) - \mu \left( \bigcap_{n=1}^{\infty} A_n \right).
\]

Comparing the above two expressions, we get the required result.

Finally, let \( A_n = [n, \infty) \), then \( \{A_n\}_{n=1}^{\infty} \) is decreasing, \( \bigcap_{n=1}^{\infty} A_n = \emptyset \), so \( \mu(\bigcap_{n=1}^{\infty} A_n) = 0 \), but \( \lim_{n \to +\infty} \mu(A_n) = +\infty \). So (ii) does not hold if we do not assume that \( \mu(A_1) < +\infty \).

8. Let \( \epsilon > 0 \) be given. By part (vi) of Theorem 3.6 (the regularity property), we can find an open set \( U \supset A \) such that

\[
\mu(U) < \mu(A) + \epsilon.
\]

Now \( U = A \cup (U \setminus A) \) (disjoint union). By part (iii) of Theorem 3.6 (countable additivity),

\[
\mu(A) + \mu(U \setminus A) = \mu(U) < \mu(A) + \epsilon.
\]
Cancelling $\mu(A)$ gives $\mu(U \setminus A) < \epsilon$, as required.

9. (a) Let $Q = \mathbb{Q} \cap [0, 1]$. Since $Q$ is countable, we know that $\mu(Q) = 0$. We have $[0, 1] = X \cup Q$ (disjoint union), so, by countable additivity,

$$1 = \mu([0, 1]) = \mu(X) + \mu(Q) = \mu(X),$$

are required.

(b) Suppose $Y \neq [0, 1]$ and write $Z = [0, 1] \setminus Y \neq \emptyset$. Then

$$1 = \mu([0, 1]) = \mu(Y) + \mu(Z) = 1 + \mu(Z),$$

so $\mu(Z) = 0$. On the other hand, $Z$ is open (relative to $[0, 1]$). If $x \in Z$, we can find $\delta > 0$ such that $(x - \delta, x + \delta) \subset Z$. This gives

$$\mu(Z) \geq \mu((x - \delta, x + \delta)) = 2\delta > 0.$$

Thus we have a contradiction and so $Y = [0, 1]$. 

6