MATH31011/MATH41011/MATH61011: FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Solution Sheet 3

1. Let $E \subset \mathbb{R}$ be a countable set. Assume E is countably infinite – the finite case is even easier. Then we can write $E = \{x_n\}_{n=1}^{\infty}$. Choose $\epsilon > 0$. Then

$$\left\{ \left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}} \right) \right\}_{n=1}^{\infty}$$

is a countable collection of open intervals such that

$$E \subset \bigcup_{n=1}^{\infty} \left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}} \right)$$

and

$$\sum_{n=1}^{\infty} l \left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}} \right) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Thus, E is a null set.

Note To make things fit the definition of a null set exactly, we could use intervals around x_n of length $\frac{1}{2^{n+1}}$ to make the sum of the lengths $\frac{\epsilon}{2}$ and thus *strictly* less than ϵ as required by the definition in the Notes. However, the above clearly suffices.

2. We start with the unit interval

$$F_0 = [0, 1].$$

Now define a new set

$$F_1 = F_0 \backslash (2/5, 3/5) = [0, 2/5] \cup [3/5, 1],$$

i.e., we obtain F_1 by deleting the open middle fifth of F_0. Next we obtain a new set F_2 by deleting the open middle fifths of each of the intervals making up F_1,

$$F_2 = \left[0, \frac{4}{25}\right] \cup \left[\frac{6}{25}, \frac{2}{5}\right] \cup \left[\frac{3}{5}, \frac{19}{25}\right] \cup \left[\frac{21}{25}, 1\right].$$

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Continue in this way to obtain sets F_n, $n \geq 0$, where F_n consists of 2^n disjoint closed intervals of length $(2/5)^n$, formed by deleting the middle thirds of the intervals making up F_{n-1}. The Middle Fifth Cantor set is defined to be the intersection of these sets:

$$F = \bigcap_{n=1}^{\infty} F_n.$$

Let $\{F_n^1, \ldots, F_n^{2^n}\}$ denote the 2^n intervals making up F_n. Enlarge these to open intervals $U_n^1, \ldots, U_n^{2^n}$ by setting

$$U_n^i = \left(a - \frac{\delta^n}{2}, b + \frac{\delta^n}{2}\right),$$

where $F_n^i = [a, b]$. Then

$$l(U_n^i) = \left(\frac{2}{5}\right)^n + \delta^n < \left(\frac{2}{5} + \delta\right)^n.$$

Now choose $\delta > 0$ sufficiently small that

$$\frac{2}{5} + \delta < \frac{1}{2},$$

i.e. take $\delta < 1/2 - 2/5 = 1/10 - \delta - 1/11$ will do.

Given $\epsilon > 0$, choose n sufficiently large that

$$2^n \left(\frac{2}{5} + \delta\right)^n < \epsilon.$$

Then

$$F \subset \bigcup_{i=1}^{2^n} U_n^i$$

and

$$\sum_{i=1}^{2^n} l(U_n^i) < \sum_{i=1}^{2^n} \left(\frac{2}{5} + \delta\right)^n = 2^n \left(\frac{2}{5} + \delta\right)^n < \epsilon.$$

Thus, F is a null set.

3.(a) Clearly $\emptyset \in \mathcal{A}$. If $A \in \mathcal{A}$ then either A is countable or $[0,1]\backslash A$ is countable. So $[0,1]\backslash A$ is countable or $[0,1]\backslash([0,1]\backslash A) = A$ is countable. Thus $[0,1]\backslash A \in \mathcal{A}$. Finally, suppose $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$. There are two possibilities:

(i) all the sets A_j are countable. Then $\bigcup_{j=1}^{\infty} A_j$ is countable and so $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

(ii) at least one of the sets A_j is uncountable. Call one of the uncountable ones A^*. Since $A^* \in \mathcal{A}$, we have that $[0,1]\backslash A^*$ is countable. Then, since

$$[0,1]\backslash \left(\bigcup_{j=1}^{\infty} A_j\right) \subset [0,1]\backslash A^*,$$

we see that $[0,1]\backslash \left(\bigcup_{j=1}^{\infty} A_j\right)$ is countable and so $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

(b) Clearly $\emptyset \in \mathcal{A}$. If $A \in \mathcal{A}$ then either A is a null set or $[0,1]\backslash A$ is a null set. So $[0,1]\backslash A$ is a null set or $[0,1]\backslash([0,1]\backslash A) = A$ is a null set. Thus $[0,1]\backslash A \in \mathcal{A}$. Finally, suppose $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$. There are two possibilities:

(i) all the sets A_j are null sets. Choose $\epsilon > 0$. Then, for each j, we can find a countable collection of open intervals $\{I_n^j\}$ such that

$$A_j \subset \bigcup_n I_n^j \quad \text{and} \quad \sum_n l(I_n^j) < \frac{\epsilon}{2^j}.$$

We have

$$\bigcup_{j=1}^{\infty} A_j \subset \bigcup_{j=1}^{\infty} \bigcup_n I_n^j$$

and

$$\sum_{j=1}^{\infty} \sum_n l(I_n^j) < \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon.$$

Thus $\bigcup_{j=1}^{\infty} A_j$ is a null set and so $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

(ii) at least one of the sets A_j is not a null set. Call one of the non-null ones A^*. Since $A^* \in \mathcal{A}$, we have that $[0,1]\backslash A^*$ is a null set. Then, since

$$[0,1]\backslash \left(\bigcup_{j=1}^{\infty} A_j \right) \subset [0,1]\backslash A^*,$$

we see that $[0,1]\backslash \left(\bigcup_{j=1}^{\infty} A_j \right)$ is a null set and so $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

4. We have $\emptyset \in \mathcal{A}_\alpha$ for each α, so $\emptyset \in \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$. Suppose that $E \in \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$. Then $E \in \mathcal{A}_\alpha$, for each α, and so $X\backslash E \in \mathcal{A}_\alpha$, for each α. Thus, $X\backslash E \in \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$. Finally, suppose that $\{E_j\}_{j=1}^{\infty} \subset \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$. Then $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}_\alpha$, for each α, and so $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}_\alpha$, for each α. Thus, $\bigcup_{j=1}^{\infty} E_j \in \bigcap_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha$.

5. Clearly $\emptyset \in \mathcal{A}(X)$. Suppose that $E \in \mathcal{A}(X)$. Then $E \in \mathcal{A}$ and $E \subset X$, so $X\backslash E = X \cap \mathbb{R}\backslash E \in \mathcal{A}$. Since $X\backslash E \subset X$, we have $X\backslash E \in \mathcal{A}(X)$. Finally, suppose that $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}(X)$. Then $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$, so $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$. Furthermore, each $E_j \subset X$, so $\bigcup_{j=1}^{\infty} E_j \subset X$. Thus, $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}(X)$.

6. *First, an argument that shows an open set is a countable union of open intervals (but doesn't show disjointness.)* Let $U \subset \mathbb{R}$ be an open set. Consider the set of open intervals (a,b) with $a, b \in \mathbb{Q}$. This is a countable collection since $\mathbb{Q} \times \mathbb{Q}$ is countable. Since U is open, if $x \in U$ then $(x-\delta, x+\delta) \subset U$, for some $\delta > 0$. We may then choose rationals a, b with $x - \delta < a < x < b < x + \delta$, so that $x \in (a,b) \subset U$. Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ denote the countable set of open intervals which arise in this way. By construction,

$$U = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

a countable union.

Now a solution to the whole question. Let $U \subset \mathbb{R}$ be an open set. For each $x \in U$, define an open interval $O(x)$ containing x by

$$O(x) = (x - \delta_1(x), x + \delta_2(x)),$$

where

$$\delta_1(x) = \sup\{\delta : (x - \delta, x] \subset U\},$$

$$\delta_2(x) = \sup\{\delta : [x, x + \delta) \subset U\}.$$

Clearly,

$$U \subset \bigcup_{x \in U} O(x).$$

Now suppose that $O(x)$ is not a subset of U. Then there exists $z \notin U$ such that

$$x - \delta_1(x) < z < x \quad \text{or} \quad x < z < x + \delta_2(x).$$

This contradicts the definition of $\delta_1(x)$ or $\delta_2(x)$. Thus, $O(x) \subset U$ and so

$$\bigcup_{x \in U} O(x) \subset U.$$

Hence

$$\bigcup_{x \in U} O(x) = U.$$

If $x, y \in U$ and $O(x) \cap O(y) \neq \emptyset$ but $O(x) \neq O(y)$ then $O(x) \cup O(y)$ is an open interval containing x, contained in U and strictly larger than $O(x)$. This contradicts the definition of $\delta_1(x)$ or $\delta_2(x)$. Thus, if $O(x) \cap O(y) \neq \emptyset$ then $O(x) = O(y)$. This shows that

$$\bigcup_{x \in U} O(x) = U$$

is a union of disjoint open intervals.

Since each $O(x)$ is an open interval, it contains a rational number $r \in \mathbb{Q}$. By the argument about, $O(r) = O(x)$. Thus, because $\mathbb{Q}$ is countable,

$$U = \bigcup_{r \in U \cap \mathbb{Q}} O(r)$$

is a *countable* union of disjoint open intervals.

7.(i) Let $B_1 = A_1$ and $B_n = A_n \backslash A_{n-1}$ for $n > 1$. Then $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, $\bigcup_{i=1}^{\infty} B_i = A_n$ and $\{B_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets. Then

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right)\\ &= \sum_{n=1}^{\infty} \mu(B_n)\\ &= \lim_{n\to+\infty} \sum_{i=1}^{n} \mu(B_i)\\ &= \lim_{n\to+\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right)\\ &= \lim_{n\to+\infty} \mu(A_n).\end{aligned}$$

(ii) Since $\{A_n\}_{n=1}^{\infty}$ is decreasing, the sequence $A'_n = A_1 \backslash A_n$ is increasing, so by (i),

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A'_n\right) &= \lim_{n\to+\infty} \mu(A'_n)\\ &= \lim_{n\to+\infty} (\mu(A_1) - \mu(A_n)) \quad \text{(since } \mu(A_1) \text{ is finite)}\\ &= \mu(A_1) - \lim_{n\to+\infty} \mu(A_n).\end{aligned}$$

But

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A'_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} (A_1 \backslash A_n)\right)\\ &= \mu\left(A_1 \backslash \bigcap_{n=1}^{\infty} A_n\right)\\ &= \mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right).\end{aligned}$$

Comparing the above two expressions, we get the required result.

Finally, let $A_n = [n, \infty)$, then $\{A_n\}_{n=1}^{\infty}$ is decreasing, $\bigcap_{n=1}^{\infty} A_n = \emptyset$, so $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$, but $\lim_{n\to+\infty} \mu(A_n) = +\infty$. So (ii) does not hold if we do not assume that $\mu(A_1) < +\infty$.

8. Let $\epsilon > 0$ be given. By part (vi) of Theorem 3.6 (the regularity property), we can find an open set $U \supset A$ such that

$$\mu(U) < \mu(A) + \epsilon.$$

Now $U = A \cup (U \backslash A)$ (disjoint union). By part (iii) of Theorem 3.6 (countable additivity),

$$\mu(A) + \mu(U \backslash A) = \mu(U) < \mu(A) + \epsilon.$$

Cancelling $\mu(A)$ gives $\mu(U\backslash A) < \epsilon$, as required.

9.(a) Let $Q = \mathbb{Q} \cap [0,1]$. Since Q is countable, we know that $\mu(Q) = 0$. We have $[0,1] = X \cup Q$ (disjoint union), so, by countable additivity,

$$1 = \mu([0,1]) = \mu(X) + \mu(Q) = \mu(X),$$

are required.

(b) Suppose $Y \neq [0,1]$ and write $Z = [0,1]\backslash Y \neq \emptyset$. Then

$$1 = \mu([0,1]) = \mu(Y) + \mu(Z) = 1 + \mu(Z),$$

so $\mu(Z) = 0$. On the other hand, Z is open (relative to $[0,1]$). If $x \in Z$, we can find $\delta > 0$ such that $(x - \delta, x + \delta) \subset Z$. This gives

$$\mu(Z) \geq \mu((x - \delta, x + \delta)) = 2\delta > 0.$$

Thus we have a contradiction and so $Y = [0,1]$.