MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Solution Sheet 2

1. (i) Since E is countable, there exists a surjection $\phi : \mathbb{N} \to E$. Then $g = f \circ \phi$ is a surjection from $\mathbb{N}$ to F and hence F is countable.

(ii) Suppose that F is countable and that $E \subset F$. The case of $E = \emptyset$ is trivial so assume $E \neq \emptyset$. Hence $F \neq \emptyset$ so let $\phi : \mathbb{N} \to F$ be a surjection. Let $\mathbb{N}' = \{n \in \mathbb{N} : \phi(n) \in E\} = \phi^{-1}(E)$. Then $\mathbb{N}'$ is a subset of $\mathbb{N}$ and $\phi|_{\mathbb{N}'} : \mathbb{N}' \to E$ is a surjection. Let e be an element of E and define $f : \mathbb{N} \to E$ by

$$f(n) = \begin{cases} \phi(n) & \text{if } n \in \mathbb{N}' \\ e & \text{otherwise} \end{cases}$$

f is a surjection so E is also countable.

2. Let

$$x = \sum_{n=1}^{\infty} a_n 10^{-n}, \quad y = \sum_{n=1}^{\infty} b_n 10^{-n},$$

with $a_n, b_n \in \{0, 1, 2, \ldots, 9\}$, for all $n \geq 1$. Assume that $x = y$. If we have $a_n = b_n$ for all $n \in \mathbb{N}$ then there is nothing to prove so let m be the least natural number for which $a_m \neq b_m$ and assume first that $a_m > b_m$. Then

$$x - y = (a_m - b_m) \cdot 10^{-m} + \sum_{n=m+1}^{\infty} (a_n - b_n) 10^{-n}$$

so

$$(a_m - b_m) \cdot 10^{-m} - \sum_{n=m+1}^{\infty} 9 \cdot 10^{-n} \leq x - y$$

with the inequality strict unless all the b_n are 9 and all the a_n are 0, for $n > m$. But $\sum_{n=m+1}^{\infty} 9 \cdot 10^{-n} = 9 \cdot 10^{-(m+1)} \frac{1}{1-\frac{1}{10}} = 10^{-m}$ and hence $x = y$ is possible only when $a_m = b_m + 1$ and $(\forall n > m\, (a_n = 0 \wedge b_n = 9))$ as required.
The case of $b_m > a_m$ is analogous with the roles of x and y swapped.

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3. Suppose, for a contradiction, that $X^{\mathbb{N}}$ is countable. Then we may write $X^{\mathbb{N}} = \{y^{(n)} : n \in \mathbb{N}\}$. But each $y^{(n)}$ is a sequence, so

$$y^{(n)} = (y_m^{(n)})_{m=1}^{\infty}, \quad \text{where } y_m^{(n)} \in \{0,1\}.$$

Define a sequence $y = (y_m)_{m=1}^{\infty}$ by the rule

$$y_m = \begin{cases} 0 & \text{if } y_m^{(m)} = 1 \\ 1 & \text{if } y_m^{(m)} = 0 \end{cases}.$$

Then $y \in X^{\mathbb{N}}$ but y disagrees with $y^{(m)}$ at the mth place, for each $m \in \mathbb{N}$. This contradiction shows that $X^{\mathbb{N}}$ is uncountable.

4.(a) We have

$$\begin{aligned} m = \sup E &\iff m \in \mathcal{U}(E) \text{ and } \forall u \in \mathcal{U}(E),\, m \le u \\ &\iff m \in \mathcal{U}(E) \text{ and } (a < m \implies a \notin \mathcal{U}(E)) \\ &\iff m \in \mathcal{U}(E) \text{ and } (\epsilon > 0 \implies (m-\epsilon) \notin \mathcal{U}(E)) \\ &\iff m \in \mathcal{U}(E) \text{ and } \forall \epsilon > 0\, \exists x \in E \text{ such that } m - \epsilon < x. \end{aligned}$$

(b) We have

$$\begin{aligned} l = \inf E &\iff m \in \mathcal{L}(E) \text{ and } \forall v \in \mathcal{L}(E),\, l \ge v \\ &\iff l \in \mathcal{L}(E) \text{ and } (a > l \implies a \notin \mathcal{L}(E)) \\ &\iff l \in \mathcal{L}(E) \text{ and } (\epsilon > 0 \implies (l+\epsilon) \notin \mathcal{L}(E)) \\ &\iff l \in \mathcal{L}(E) \text{ and } \forall \epsilon > 0 \exists x \in E \text{ such that } x < l + \epsilon. \end{aligned}$$

5.(a) Let $a_n = \sup_{k \ge n} x_k$. Note that a_n can be $+\infty$. Also, for $m < n$ we have $a_m \ge a_n$ so $\{a_n\}$ is a decreasing sequence and as such it has a limit (which can be real, or $-\infty$, or $+\infty$ if all the a_n are $+\infty$.

Assume first that $lim_{n\to\infty} a_n = a \in \mathbb{R}$. Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that for $n \ge N$ we have $|a - a_n| < \frac{\epsilon}{2}$, so

$$a - \frac{\epsilon}{2} \;<\; \sup_{k \ge n} x_k \;<\; a + \frac{\epsilon}{2}$$

The second inequality gives that for all $n \ge N$, $x_n < a + \epsilon$ and the first inequality means that for any $n \ge N$ there is some $k \ge n$ such that $x_k > a - \epsilon$, so there are infinitely many natural numbers k for which $x_k > a - \epsilon$. Hence $a = \limsup_{n\to+\infty} x_n$.

If $\lim_{n\to\infty} a_n = +\infty$ then for each $M > 0$ there is $N \in \mathbb{N}$ such that for all $n \ge N$ we have $a_n = \sup_{k \ge n} x_k \ge M + 1$ and hence for some $k \ge n$ also $x_k > M$, so $x_k > M$ for infinitely many values of k and hence $\limsup_{n\to+\infty} x_n = \infty$.

Finally, if $\lim_{n\to\infty} a_n = -\infty$ then for each $M > 1$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n = \sup_{k\geq n} x_k \leq -M$ so we also have that for all $n \geq N$, $x_n \leq -M$ and hence $\lim_{n\to\infty} x_n = -\infty = \limsup_{n\to+\infty} x_n$.

(b) Let $b_n = \inf_{k\geq n} x_k$. Note that b_n can be $-\infty$. Also, for $m < n$ we have $b_m \leq b_n$ so $\{b_n\}$ is an increasing sequence and as such it has a limit (which can be real, or $+\infty$, or $-\infty$ if all the b_n are $-\infty$).

Assume first that $lim_{n\to\infty} b_n = b \in \mathbb{R}$. Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that for $n \geq N$ we have $|b - b_n| < \frac{\epsilon}{2}$, so

$$b - \frac{\epsilon}{2} \; < \; \inf_{k\geq n} x_k \; < \; b + \frac{\epsilon}{2}$$

The first inequality gives that for all $n \geq N$, $x_n > b - \epsilon$ and the second inequality means that for any $n \geq N$ there is some $k \geq n$ such that $x_k < b + \epsilon$, so there are infinitely many natural numbers k for which $x_k < b + \epsilon$. Hence $b = \liminf_{n\to+\infty} x_n$.

If $\lim_{n\to\infty} b_n = -\infty$ then for each $M > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $b_n = \inf_{k\geq n} x_k \leq -M - 1$ and hence for some $k \geq n$ also $x_k < -M$, so $x_k < -M$ for infinitely many values of k and hence $\liminf_{n\to+\infty} x_n = -\infty$.

Finally, if $\lim_{n\to\infty} b_n = +\infty$ then for each $M > 1$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $b_n = \inf_{k\geq n} x_k \geq M$ so we also have that for all $n \geq N$, $x_n \geq M$ and hence $\lim_{n\to\infty} x_n = +\infty = \liminf_{n\to+\infty} x_n$.

6. Let $\alpha = \sup E$ and $\beta = \sup F$. Then, for all $e \in E$, $e \leq \alpha$, and, for all $f \in F$, $f \leq \beta$. So, for all $x \in E \cup F$, $x \leq \max\{\alpha, \beta\}$, i.e., $\max\{\alpha, \beta\} \in \mathcal{U}(E \cup F)$.

Let $m \in \mathcal{U}(E \cup F)$. That means that for all $x \in E \cup F$ we have $x \leq m$ and hence both $m \in \mathcal{U}(E)$ and $m \in \mathcal{U}(F)$. Since $\alpha = \sup E$ and $\beta = \sup F$, it follows that $\alpha \leq m$ and $\beta \leq m$ so $\max\{\alpha, \beta\} \leq m$. Hence

$$\max\{\alpha, \beta\} = \max\{\sup E, \sup F\} = sup(E \cup F)$$

as required.