1. (a) One can try evaluating the integral by parts but perhaps the simplest approach is to write \[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \]

Then the integral becomes

\[
I = \int_{-1}^{1} \left( \frac{e^{i(2m-1)\pi x/2} + e^{-i(2m-1)\pi x/2}}{2} \right) \left( \frac{e^{i(2n-1)\pi x/2} + e^{-i(2n-1)\pi x/2}}{2} \right) \, dx
\]

\[
= \frac{1}{4} \int_{-1}^{1} \left( e^{i(m+n-1)\pi x} + e^{i(n-m)\pi x} + e^{i(m-n)\pi x} + e^{-i(m+n-1)\pi x} \right) \, dx.
\]

Now, for \( k \in \mathbb{Z} \backslash \{0\}, \)

\[
\int_{-1}^{1} e^{\pi ikx} \, dx = \frac{e^{\pi ik} - e^{-\pi ik}}{\pi k} = \frac{2}{\pi k} \sin(k\pi) = 0,
\]

while, for \( k = 0, \)

\[
\int_{-1}^{1} e^{2\pi ikx} \, dx = \int_{-1}^{1} \, dx = 2.
\]

Thus, if \( m \neq n, \) we have \( \pm (m + n - 1), \pm (n - m) \neq 0 \) and so we get \( I = 0. \)

On the other hand, if \( m = n \) then \( \pm (n - m) = 0 \) and we get

\[
I = \frac{1}{4} (2 + 2) = 1,
\]

as required.

(b) Suppose that

\[
f(x) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{(2n-1)\pi x}{2} \right).
\]
Then, using (a),
\[
\int_{-1}^{1} f(x) \cos \left( \frac{(2m-1)\pi x}{2} \right) \, dx
= \int_{-1}^{1} \left( \sum_{n=1}^{\infty} a_n \cos \left( \frac{(2n-1)\pi x}{2} \right) \right) \cos \left( \frac{(2m-1)\pi x}{2} \right) \, dx
= \sum_{n=1}^{\infty} a_n \int_{-1}^{1} \cos \left( \frac{(2n-1)\pi x}{2} \right) \cos \left( \frac{(2n-1)\pi x}{2} \right) \, dx = a_m,
\]
as required.

(c) To show the formula, we just need to use (b) to evaluate \( a_n \) for the constant function 1. We have
\[
a_n = \int_{-1}^{1} \cos \left( \frac{(2n-1)\pi x}{2} \right) \, dx = \frac{2 \sin((2n-1)\pi x/2)}{(2n-1)\pi} \bigg|_{x=-1}^{x=1}
= \frac{2 \sin((2n-1)\pi/2)}{(2n-1)\pi} - \frac{2 \sin(-((2n-1)\pi/2))}{(2n-1)\pi}
= \frac{4 \sin((2n-1)\pi/2)}{(2n-1)\pi} = \frac{4(-1)^{n-1}}{(2n-1)\pi}
\]
(since \( \sin((2n-1)\pi/2) = 1 \) if \( n \) is odd and \( -1 \) if \( n \) is even).

(d) If we put \( x = 0 \) then \( \cos((2n-1)\pi x/2) = \cos(0) = 1 \). Thus
\[
1 = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{4}{\pi} \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \right).
\]
Rearranging gives
\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}.
\]
2. To calculate the Fourier series for \( f \), we need to work out the Fourier coefficients \( a_m \) and \( b_m \). For \( m = 0 \), we have

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \left( -\int_{-\pi}^{-\pi/2} dx + \int_{-\pi/2}^{\pi/2} dx - \int_{\pi/2}^{\pi} dx \right) = \frac{1}{\pi} \left( \frac{-\pi}{2} + \pi - \frac{-\pi}{2} \right) = 0.
\]

For \( m \geq 1 \), we have

\[
a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx = \frac{1}{\pi} \left( -\int_{-\pi}^{-\pi/2} \cos(mx) \, dx + \int_{-\pi/2}^{\pi/2} \cos(mx) \, dx - \int_{\pi/2}^{\pi} \cos(mx) \, dx \right) = \frac{1}{\pi m} (\sin(-\pi m) - \sin(-\pi m/2) + \sin(\pi m/2) - \sin(-\pi m/2) + \sin(\pi m/2) - \sin(\pi m)) = \frac{4}{\pi m} \sin(\pi m/2)
\]

(where we have used \( \sin(-\theta) = -\sin \theta \)). Thus

\[
a_m = \begin{cases} 
0 & \text{if } m \text{ is even} \\
(-1)^{n-1} & \text{if } m = 2n - 1 \text{ is odd.}
\end{cases}
\]

Also, for \( m \geq 1 \),

\[
b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx = \frac{1}{\pi} \left( -\int_{-\pi}^{-\pi/2} \sin(mx) \, dx + \int_{-\pi/2}^{\pi/2} \sin(mx) \, dx - \int_{\pi/2}^{\pi} \sin(mx) \, dx \right) = \frac{-1}{\pi m} (\cos(-\pi m) - \cos(-\pi m/2) + \cos(\pi m/2) - \cos(-\pi m/2) + \cos(\pi m/2) - \cos(\pi m))
\]
(where we have used $\cos(-\theta) = \cos \theta$). Putting these values together gives the claimed Fourier series.

3. (a) Suppose that the functions $f_n : [a, b] \to \mathbb{R}$ converge uniformly to 0. By definition, given $\epsilon > 0$, there exists $N \geq 1$ such that if $n \geq N$ then

$$|f_n(x)| < \epsilon \quad \text{for all} \ x \in [a, b].$$

Then, using standard properties of the Riemann integral, if $n \geq N$,

$$\left| \int_a^b f_n(x) \, dx \right| \leq \int_a^b |f_n(x)| \, dx \leq (b-a)\epsilon.$$

This shows that the sequence of numbers $\int_a^b f_n(x) \, dx$ converges to 0 as $n \to +\infty$.

(b) For our example, we’ll take $[a, b] = [0, 1]$. Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq 1/n \\ 2n - n^2 x & \text{if } 1/n < x \leq 2/n \\ 0 & \text{if } 2/n < x \leq 1. \end{cases}$$

(Draw a picture!) Clearly, each $f_n$ is continuous.

We need to show that $f_n$ converges pointwise to 0, i.e. that $\lim_{n \to +\infty} f_n(x) = 0$ for every $x \in [0, 1]$. First, we note that $f_n(0) = 0$ for all $n$, so $\lim_{n \to +\infty} f_n(0) = 0$. Now suppose that $x > 0$. If $n > 2/x$ then $x > 2/n$, so that $f_n(x) = 0$, i.e. $f_n(x) = 0$ for all sufficiently large $n$. Thus, $\lim_{n \to +\infty} f_n(x) = 0$.

Finally, we need to show that the numbers $\int_0^1 f_n(x) \, dx$ do not converge to 0. We calculate

$$\int_0^1 f_n(x) \, dx = \int_0^{1/n} n^2 x \, dx + \int_{1/n}^{2/n} (2n - n^2 x) \, dx$$
\[
\frac{n^2 x^2}{2} \bigg|_{x=0}^{x=1/n} + \left(2nx - \frac{n^2 x^2}{2} \right) \bigg|_{x=1/n}
\]

\[
= \frac{1}{2} + (4 - 2) - \left(2 - \frac{1}{2}\right) = 1,
\]

for all \( n \), which, of course, does not tend to 0, as \( n \to +\infty \). (Rather than calculating the integral, it is easier to spot that it is the area of a triangle with base \( 2/n \) and vertical height \( n \).)