MATH31011/MATH41011/MATH61011: Fourier Analysis and Lebesgue Integration

Solutions 1

1.(a) One can try evaluating the integral by parts but perhaps the simplest approach is to write

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Then the integral becomes

$$\begin{aligned} I &= \int_{-1}^{1} \left(\frac{e^{i(2m-1)\pi x/2} + e^{-i(2m-1)\pi x/2}}{2} \right) \left(\frac{e^{i(2n-1)\pi x/2} + e^{-i(2n-1)\pi x/2}}{2} \right) dx \\ &= \frac{1}{4} \int_{-1}^{1} \left(e^{i(m+n-1)\pi x} + e^{i(n-m)\pi x} + e^{i(m-n)\pi x} + e^{-i(m+n-1)\pi x} \right) dx. \end{aligned}$$

Now, for $k \in \mathbb{Z}\backslash\{0\}$,

$$\int_{-1}^{1} e^{\pi i k x}\, dx = \left. \frac{e^{\pi i k x}}{\pi i k} \right|_{x=-1}^{x=1} = \frac{e^{\pi i k}}{\pi i k} - \frac{e^{-\pi i k}}{\pi i k} = \frac{2}{\pi k} \sin(k\pi) = 0,$$

while, for $k = 0$,

$$\int_{-1}^{1} e^{2\pi i k x}\, dx = \int_{-1}^{1} dx = 2.$$

Thus, if $m \neq n$, we have $\pm(m+n-1), \pm(n-m) \neq 0$ and so we get $I = 0$. On the other hand, if $m = n$ then $\pm(n-m) = 0$ and we get

$$I = \frac{1}{4}(2+2) = 1,$$

as required.

(b) Suppose that

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{2} \right).$$

Then, using (a),

$$\begin{aligned}
&\int_{-1}^{1} f(x)\cos\left(\frac{(2m-1)\pi x}{2}\right)\,dx \\
&\quad= \int_{-1}^{1}\left(\sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{2}\right)\right)\cos\left(\frac{(2m-1)\pi x}{2}\right)\,dx \\
&\quad= \sum_{n=1}^{\infty} a_n \int_{-1}^{1}\cos\left(\frac{(2n-1)\pi x}{2}\right)\cos\left(\frac{(2n-1)\pi x}{2}\right)\,dx = a_m,
\end{aligned}$$

as required.

(c) To show the formula, we just need to use (b) to evaluate a_n for the constant function 1. We have

$$\begin{aligned}
a_n &= \int_{-1}^{1}\cos\left(\frac{(2n-1)\pi x}{2}\right)\,dx = \left.\frac{2\sin((2n-1)\pi x/2)}{(2n-1)\pi}\right|_{x=-1}^{x=1} \\
&= \frac{2\sin((2n-1)\pi/2)}{(2n-1)\pi} - \frac{2\sin(-(2n-1)\pi/2)}{(2n-1)\pi} \\
&= \frac{4\sin((2n-1)\pi/2)}{(2n-1)\pi} = \frac{4(-1)^{n-1}}{(2n-1)\pi}
\end{aligned}$$

(since $\sin((2n-1)\pi/2) = 1$ if n is odd and -1 if n is even).

(d) If we put $x = 0$ then $\cos((2n-1)\pi x/2) = \cos(0) = 1$. Thus

$$1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right).$$

Rearranging gives

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}.$$

2. To calculate the Fourier series for f, we need to work out the Fourier coefficients a_m and b_m. For $m = 0$, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(-\int_{-\pi}^{-\pi/2} dx + \int_{-\pi/2}^{\pi/2} dx - \int_{\pi/2}^{\pi} dx\right) = \frac{1}{\pi}\left(-\frac{\pi}{2} + \pi - \frac{\pi}{2}\right) = 0.$$

For $m \geq 1$, we have

$$\begin{aligned}
a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \\
&= \frac{1}{\pi}\left(-\int_{-\pi}^{-\pi/2} \cos(mx)\,dx + \int_{-\pi/2}^{\pi/2} \cos(mx)\,dx - \int_{\pi/2}^{\pi} \cos(mx)\,dx\right) \\
&= \frac{1}{\pi m}\left(\sin(-\pi m) - \sin(-\pi m/2) + \sin(\pi m/2) - \sin(-\pi m/2) + \sin(\pi m/2) - \sin(\pi m)\right) \\
&= \frac{4}{\pi m}\sin(\pi m/2)
\end{aligned}$$

(where we have used $\sin(-\theta) = -\sin\theta$). Thus

$$a_m = \begin{cases} 0 & \text{if } m \text{ is even} \\ (-1)^{n-1} & \text{if } m = 2n-1 \text{ is odd.} \end{cases}$$

Also, for $m \geq 1$,

$$\begin{aligned}
b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \\
&= \frac{1}{\pi}\left(-\int_{-\pi}^{-\pi/2} \sin(mx)\,dx + \int_{-\pi/2}^{\pi/2} \sin(mx)\,dx - \int_{\pi/2}^{\pi} \sin(mx)\,dx\right) \\
&= \frac{-1}{\pi m}\left(\cos(-\pi m) - \cos(-\pi m/2) + \cos(\pi m/2) - \cos(-\pi m/2) + \cos(\pi m/2) - \cos(\pi m)\right)
\end{aligned}$$

$$= \quad 0$$

(where we have used $\cos(-\theta) = \cos\theta$). Putting these values together gives the claimed Fourier series.

3.(a) Suppose that the functions $f_n : [a, b] \to \mathbb{R}$ converge uniformly to 0. By definition, given $\epsilon > 0$, there exists $N \geq 1$ such that if $n \geq N$ then

$$|f_n(x)| < \epsilon \qquad \text{for all } x \in [a, b].$$

Then, using standard properties of the Riemann integral, if $n \geq N$,

$$\left| \int_a^b f_n(x)\, dx \right| \leq \int_a^b |f_n(x)|\, dx \leq (b - a)\epsilon.$$

This shows that the sequence of numbers $\int_a^b f_n(x)\, dx$ converges to 0 as $n \to +\infty$.

(b) For our example, we'll take $[a, b] = [0, 1]$. Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq 1/n \\ 2n - n^2 x & \text{if } 1/n < x \leq 2/n \\ 0 & \text{if } 2/n < x \leq 1. \end{cases}$$

(Draw a picture!) Clearly, each f_n is continuous.

We need to show that f_n converges pointwise to 0, i.e. that $\lim_{n\to+\infty} f_n(x) = 0$ for every $x \in [0, 1]$. First, we note that $f_n(0) = 0$ for all n, so $\lim_{n\to+\infty} f_n(0) = 0$. Now suppose that $x > 0$. If $n > 2/x$ then $x > 2/n$, so that $f_n(x) = 0$, i.e. $f_n(x) = 0$ for all sufficiently large n. Thus, $\lim_{n\to+\infty} f_n(x) = 0$.

Finally, we need to show that the numbers $\int_0^1 f_n(x)\, dx$ do not converge to 0. We calculate

$$\int_0^1 f_n(x)\, dx \quad = \quad \int_0^{1/n} n^2 x\, dx + \int_{1/n}^{2/n} (2n - n^2 x)\, dx$$

$$= \left.\frac{n^2x^2}{2}\right|_{x=0}^{x=1/n} + \left.\left(2nx - \frac{n^2x^2}{2}\right)\right|_{x=1/n}^{x=2/n}$$

$$= \frac{1}{2} + (4-2) - \left(2 - \frac{1}{2}\right) = 1,$$

for all n, which, of course, does not tend to 0, as $n \to +\infty$. (Rather than calculating the integral, it is easier to spot that it is the area of a triangle with base $2/n$ and vertical height n.)