MATH41011/MATH61011:
FOURIER SERIES AND LEBESGUE INTEGRATION

Extra Reading Material for Level 4 and Level 6

Part A: Construction of Lebesgue Measure

The first part the extra material consists of the construction of the Lebesgue measure μ whose existence is asserted in Theorem 3.6:

Theorem 3.6. *There exists a unique function $\mu : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ such that*

(i) *if I is an interval then $\mu(I) = l(I)$ (length property);*
(ii) *for every $x \in \mathbb{R}$, $\mu(x + E) = \mu(E)$ (translation invariance);*
(iii) *if $\{E_j\}_{j=1}^{\infty}$ is a countable collection of disjoint sets (in $\mathcal{M}$) then*

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

(countable additivity);
(iv) *$E \subset F \implies \mu(E) \leq \mu(F)$ (montonicity);*
(v) *if $A \in \mathcal{N}$ then $\mu(A) = 0$ and, conversely, if $\mu(A) = 0$ then $A \in \mathcal{N}$ (null sets property);*
(vi) *if $A \in \mathcal{M}$ then $\mu(A) = \inf\{\mu(U) : U$ is open and $A \subset U\}$ (regularity).*

Recall that $\mathcal{M}$ denotes the smallest σ-algebra containing $\mathcal{O} \cup \mathcal{N}$ and that sets in $\mathcal{M}$ are called (Lebesgue) measurable. If I is an interval then $\mathcal{M}(I)$ will denote the measurable subsets of I.

Lebesgue Outer Measure. The first step in the construction is to introduce the so-called *Lebesgue outer measure*, μ^*. This will be defined for all subsets of $\mathbb{R}$ and will satisfy properties (i), (ii), (iv), (v) and (vi) in Theorem 3.6 but will satisfy a condition which is weaker than countable additivity.
Let $\{I_j\}$ be a countable collection of intervals. For $E \subset \mathbb{R}$, we say that $\{I_j\}$ is a cover of E if $E \subset \bigcup_j I_j$.

Definition Let $E \subset \mathbb{R}$. Its Lebesgue outer measure $\mu^*(E)$ is defined by

$$\mu^*(E) = \inf\left\{\sum_j l(I_j) : \{I_j\} \text{ is a cover of } E \text{ by intervals}\right\}.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(Note that we may have $\mu^*(E) = +\infty$.)

Exercise 1 *Show that we obtain the same value for $\mu^*(E)$ if we only allow covers $\{I_j\}$ consisting of open intervals.*

With the above definition, it is easy to establish some basic properties of μ^*.

Proposition A.1. *The Lebegsue outer measure μ^* satisfies the following.*

(i) $\mu^*(E) \geq 0$ *for all* $E \subset \mathbb{R}$;
(ii) $\mu^*(\emptyset) = 0$;
(iii) $E \subset F \implies \mu^*(E) \leq \mu^*(F)$ *(monotonicity)*;
(iv) $\mu^*(\{x\}) = 0$ *for all* $x \in \mathbb{R}$.

Proof.
(i) For any collection of intervals $\{I_j\}$, $\sum_j l(I_j) \geq 0$, so the infimum in the definition of $\mu^*(E)$ is ≥ 0.

(ii) Since $\emptyset$ is a subset of any set, $\{[-\epsilon/2, \epsilon/2]\}$ is a cover of $\emptyset$, for any $\epsilon > 0$. Hence

$$\begin{aligned}\mu^*(\emptyset) &= \inf \left\{ \sum_j l(I_j) \,:\, \{I_j\} \text{ is a cover of } \emptyset \text{ by intervals} \right\} \\ &\leq l([-\epsilon/2, \epsilon/2]) = \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\mu^*(\emptyset) = 0$.

(iii) If $E \subset F$ then every cover of F is also a cover of E, so the infimum in the definition of $\mu^*(E)$ is taken over a larger collection than that in the definition of $\mu^*(F)$. Hence $\mu^*(E) \leq \mu^*(F)$.

(iv) This is nearly the same as the proof that $\mu^*(\emptyset) = 0$. Simply observe that, for any $\epsilon > 0$, $\{[x - \epsilon/2, x + \epsilon/2]\}$ is a cover of $\{x\}$. □

Next, we show that μ^* satisfies the null sets property.

Proposition A.2. *For a set $A \subset \mathbb{R}$, $A \in \mathcal{N}$ if and only if $\mu^*(A) = 0$.*

Proof. Suppose that $A \in \mathcal{N}$. Then, by the definition of null sets, given $\epsilon > 0$, there is a countable collection of open intervals $\{I_j\}$ such that $A \subset \bigcup_j I_j$ – so that $\{I_j\}$ is a cover of A and $\sum_j l(I_j) < \epsilon$. Thus, by the definition of $\mu^*(A)$ as an infimum,

$$\mu^*(A) \leq \sum_j l(I_j) < \epsilon.$$

Since this holds for all $\epsilon > 0$, we get $\mu^*(A) = 0$.
Now suppose $\mu^*(A) = 0$. Then, by the definition of infimum, given $\epsilon > 0$, there is an open cover $\{I_j\}$ of A such that

$$\sum_j l(I_j) < \mu^*(A) + \epsilon = \epsilon.$$

This shows that A is a null set. □

To end this subsection, we shall show that μ^* satisfies the translation invariance property.

Proposition A.3. *For all $E \subset \mathbb{R}$ and all $x \in \mathbb{R}$, we have $\mu^*(x+E) = \mu^*(E)$.*

Exercise 2 *Prove Proposition A.3.*

The Length Property. Next we need to show that μ^* really is a generalization of length by proving that μ^* satisfies the length property. This is much harder than the fairly easy results we proved above. It uses the following theorem, the proof of which makes a fairly challenging exercise.

Heine-Borel Theorem. *Suppose that I is a closed and bounded (or, as we sometimes say, compact) interval in $\mathbb{R}$. Then every open cover of I has a finite subcover. More formally: if $\{J_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of open intervals which cover I then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{A}$ such that $I \subset J_{\alpha_1} \cup J_{\alpha_2} \cup \cdots \cup J_{\alpha_n}$. (In this result, the collection $\mathcal{A}$ does not have to be countable. However, the hypothesis that the intervals J_α are open is absolutely crucial.)*

Exercise 3 *Prove the Heine-Borel Theorem.*

Hint: Consider

$$S = \{x \in [a,b] : [a,x] \text{ can be covered by a finite number of the } J_\alpha\}.$$

Show that $S \neq \emptyset$ and let $c = \sup S$. Show that

(i) $c = b$; and
(ii) $b \in S$.

Proposition A.4. *If $I \subset \mathbb{R}$ is an interval then $\mu^*(I) = l(I)$.*

Proof. First observe that, since $\{I\}$ is a cover of I, it is clear that

$$\mu^*(I) \leq l(I).$$

Now we have to show the reverse inequality: this is considerably harder. We shall first consider the case where I is a closed and bounded interval $[a,b]$. By Exercise 1, we know that $\mu^*(I)$ is the infimum of $\sum_j l(J_j)$ taken over all covers of I by open intervals. However, the Heine-Borel Theorem implies that every such cover has a *finite* subcover, $\{J_{j_1}, \ldots, J_{j_n}\}$ say. Clearly,

$$\sum_{i=1}^{n} l(I_{j_i}) \leq \sum_j l(I_j),$$

so we obtain that $\mu^*(I)$ is the infimum of $\sum_j l(J_j)$ taken over all covers of I by a finite collection of open intervals. Thus, we only need to consider covers of I consisting of a finite number of open intervals. We shall use induction on the number of elements in the cover to show that if $\{J_1, \ldots, J_n\}$ is a cover of a closed interval I then

$$l(I) \leq \sum_{i=1}^{n} l(J_i).$$

If a closed interval I is covered by one interval $\{J_1\}$ then, by monotonicity, $l(I) \leq l(J_1)$. Suppose, as our inductive hypothesis, that if a closed interval I is covered by $m \leq n$ intervals $\{J_1, \ldots, J_m\}$ then

$$l(I) \leq \sum_{i=1}^{m} l(J_i).$$

Now suppose that I is covered by $\{J_1, \ldots, J_{n+1}\}$ but that no n of these intervals cover I. (If they do the result is trivial.) Consider $I \backslash J_{n+1}$: this consists of either one or two disjoint closed intervals. Let us suppose there are two – if there is only one the argument is simpler – and call them I' and I''. None of the intervals J_i, $i = 1, \ldots, n$, can intersect both I' and I'', for otherwise this J_i would contain $I \cap J_{n+1}$ and hence $\{J_1, \ldots, J_n\}$ would be a cover of I. Therefore, some of the intervals $J_1, \ldots, J_n$ form a cover of I' and the rest form a cover of I''. Hence, by the inductive hypothesis, we have

$$l(I') + l(I'') \leq \sum_{i=1}^{n} l(J_i),$$

which gives us

$$\begin{aligned} l(I) &= l(I') + l(I'') + l(I \cap J_{n+1}) \\ &\leq l(I') + l(I'') + l(J_{n+1}) \\ &\leq \sum_{i=1}^{n+1} l(J_i), \end{aligned}$$

as required. This completes the induction.

We have shown that for any cover of a closed interval I by a finite number of open intervals $\{J_1, \ldots, J_n\}$, we have

$$l(I) \leq \sum_{i=1}^{n} l(J_i).$$

Taking the infimum over all such covers gives

$$l(I) \leq \mu^*(I),$$

so that $\mu^*(I) = l(I)$.

Now suppose that $I = (a, b)$ is an open bounded interval. Then, if $\epsilon > 0$ is sufficiently small, $[a + \epsilon/2, b - \epsilon/2]$ is a closed interval contained in I. By monotonicity,

$$\begin{aligned} \mu^*(I) &\geq \mu^*([a + \epsilon/2, b - \epsilon/2]) \\ &= l([a + \epsilon/2, b - \epsilon/2]) \\ &= b - a - \epsilon = l(I) - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\mu^*(I) \geq l(I)$. Combined with the first paragraph, this gives $\mu^*(I) = l(I)$.

Similar arguments deal with half-open bounded intervals. Finally, any unbounded interval contains bounded subintervals of arbitrary lengths so the result follows by monotonicity of μ^*. □

Exercise 4 Give an example of a subset $A \subset [0,1]$ such that $\mu^*(A) = 0$ but such that if $\{U_1, U_2, \dots, U_n\}$ is a finite cover of A by open intervals then $\sum_{i=1}^{n} l(U_i) \geq 1$.

Regularity.

Proposition A.5. *If $A \subset \mathbb{R}$ then*

$$\mu^*(A) = \inf\{\mu^*(U) : U \textit{ is open and } A \subset U\}.$$

Exercise 5. Prove Proposition A.5. *Hint:* use Exercise 1.

Countable Subadditivity. Next, we show that μ^* satisfies a weakened version of countable additivity. This is called *countable subadditivity.*

Proposition A.6. *Let $\{E_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{R}$. (Note that we do not assume that they are disjoint.) Then*

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

Proof. If any of the E_i's has $\mu^*(E_i) = +\infty$ then the result is immediate, so we shall assume that $\mu^*(E_i) < +\infty$ for all i.
Let $\epsilon > 0$. For each i, we can, by the definition of $\mu^*(E_i)$, choose a cover $\{I_j^{(i)}\}_{j=1}^{\infty}$ of E_i by intervals so that

$$\mu^*(E_i) > \sum_{j=1}^{\infty} l(I_j^{(i)}) - \frac{\epsilon}{2^i}.$$

Then $\bigcup_{i=1}^{\infty}\{I_j^{(i)}\}_{j=1}^{\infty}$ is a cover of $\bigcup_{i=1}^{\infty} E_i$ and so

$$\begin{aligned}
\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} l(I_j^{(i)}) \\
&\leq \sum_{i=1}^{\infty}\left(\mu^*(E_i) + \frac{\epsilon}{2^i}\right) \\
&= \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary the required inequality follows. □

Exercise 6 (a) Show that, if $N \in \mathcal{N}$ and $A \subset \mathbb{R}$ then $\mu^*(A \cup N) = \mu^*(A)$.

(b) Recall that, for $E, F \subset \mathbb{R}$, their symmetric difference is defined by $E \triangle F = (E \backslash F) \cup (F \backslash E)$. Show that if $E \triangle F \in \mathcal{N}$ then $\mu^*(E) = \mu^*(F)$.

Exercise 7 Suppose that $\{I_j\}$ is a countable collection of disjoint open intervals. Prove that

$$\mu^* \left(\bigcup_j I_j \right) = \sum_j l(I_j).$$

Exercise 8 Let $\{I_j\}$ be as in the previous exercise and let A be a subset of $\mathbb{R}$. Show that

$$\mu^* \left(A \cap \bigcup_j I_j \right) = \sum_j \mu^*(A \cap I_j).$$

The collection $\mathcal{M}_0$. We will see later on (in Part B) that it is possible to find sets $A, B \subset \mathbb{R}$ such that $A \cap B = \emptyset$ and

$$\mu^*(A) + \mu^*(B) \neq \mu^*(A \cup B),$$

so countable additivity fails in a very dramatic way. Our aim is to find a "big" σ-algebra on which this cannot happen. Here "big" will mean that it includes all Borel sets.
We begin with some motivation. Suppose that $\mathcal{A}$ is a σ-algebra that contains all Borel sets and suppose that $\mu : \mathcal{A} \to \mathbb{R}^+ \cup \{+\infty\}$ is a function which satisfies the length property, monotonicity *and* countable additivity. The next result will give us some estimates that such a function μ must satisfy.

Lemma A.7. *Suppose that $\mu : \mathcal{A} \to \mathbb{R}^+ \cup \{+\infty\}$ is a function which satisfies the length property, monotonicity and countable additivity. Then, for any interval $I = [a, b]$ and any set $A \in \mathcal{A}$,*

$$l(I) - \mu^*(A^c \cap I) \leq \mu(A \cap I) \leq \mu^*(A \cap I).$$

Proof. First we shall show the upper bound. Suppose that $\{I_j\}$ is a cover of $A \cap I$ by open intervals. Then $\{I_j\}$ is contained in $\mathcal{A}$ and

$$\mu(A \cap I) \leq \mu \left(\bigcup_j I_j \right) \leq \sum_j \mu(I_j) = \sum_j l(I_j),$$

where the first inequality follows from monotonicity and the second inequality follows from countable additivity. Taking the infimum over all covers of $A \cap I$ by open intervals we obtain

$$\mu(A \cap I) \leq \mu^*(A \cap I).$$

Thus the upper bound holds.

Since $A^c \in \mathcal{A}$, as well, the same argument also shows that

$$\mu(A^c \cap I) \leq \mu^*(A^c \cap I).$$

Futhermore, $A \cap I$ and $A^c \cap I$ are disjoint, so countable additivity gives

$$\mu(A \cap I) + \mu(A^c \cap I) = \mu((A \cap I) \cup (A^c \cap I)) = \mu(I) = l(I),$$

where we have used the length property. Rearranging this and using the inequality for $A^c \cap I$ gives

$$\mu(A \cap I) = l(I) - \mu(A^c \cap I) \geq l(I) - \mu^*(A^c \cap I),$$

so the lower bound holds. $\square$

This result tells us that if $\mathcal{A}$ and μ satisfy the properties we want then we must have

$$l(I) - \mu^*(A^c \cap I) \leq \mu(A \cap I) \leq \mu^*(A \cap I)$$

for all sets $A \in \mathcal{A}$ and all intervals I. If we had

$$l(I) - \mu^*(A^c \cap I) = \mu^*(A \cap I)$$

then the value of $\mu(A \cap I)$ would be determined by this. We now make an educated guess – which we will need to jusify – that we should focus on sets for which the equality $l(I) - \mu^*(A^c \cap I) = \mu^*(A \cap I)$ holds.

Definition We will write $\mathcal{M}_0$ for the collection of sets $A \subset \mathbb{R}$ which have have the property that for every set $X \subset \mathbb{R}$,

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) = \mu^*(X).$$

Since μ^* is subadditive, the definition can be relaxed to the requirement that for every $X \subset \mathbb{R}$,

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) \leq \mu^*(X).$$

Often, we will call the sets X "test sets" to indicate that they are being used to test whether a given set A is contained in $\mathcal{M}_0$.

Exercise 9 Let $\mathcal{S}$ denote the collection of all subsets A of $\mathbb{R}$ such that, for every interval $J = (a, b)$,

$$\mu^*(A \cap J) + \mu^*(A^c \cap J) = \mu^*(J).$$

(a) Show that if $A \in \mathcal{S}$ and U is a bounded open subset of $\mathbb{R}$ then

$$\mu^*(A \cap U) + \mu^*(A^c \cap U) = \mu^*(U).$$

(b) Use (a) to show that if $A \in \mathcal{S}$ and X is any bounded subset of $\mathbb{R}$ then

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) = \mu^*(X).$$

Proposition A.8. *$\mathcal{N} \subset \mathcal{M}_0$.*

Exercise 10 Prove Proposition A.8.

The next result shows that if we restrict μ^* to $\mathcal{M}_0$ then it satisfies countable additivity.

Theorem A.9. *Let $\{A_j\}_{j=1}^{\infty}$ be a countable collection of disjoint sets in $\mathcal{M}_0$. Then*

$$\mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} \mu^*(A_j).$$

Proof. Consider the first n sets $A_1, \ldots, A_n$. Set $A = A_1$ and $X = A_1 \cup A_2 \cup \cdots \cup A_n$ in the definition of $\mathcal{M}_0$. Since $A_1 \in \mathcal{M}_0$, we obtain

$$\mu^*(A_1) + \mu^*(A_2 \cup \cdots \cup A_n) = \mu^*(A_1 \cup \cdots \cup A_n).$$

Repeating the argument with $A = A_2$ and $X = A_2 \cup \cdots \cup A_n$, we obtain

$$\mu^*(A_2) + \mu^*(A_3 \cup \cdots \cup A_n) = \mu^*(A_2 \cup \cdots \cup A_n),$$

so that

$$\mu^*(A_1) + \mu^*(A_2) + \mu^*(A_3 \cup \cdots \cup A_n) = \mu^*(A_1 \cup \cdots \cup A_n).$$

By repeating this argument a finite number of times, we get

$$\mu^* \left(\bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \mu^*(A_j).$$

Since, for any $n \geq 1$, we have $\bigcup_{j=1}^{n} A_j \subset \bigcup_{j=1}^{\infty} A_j$, we obtain the inequality

$$\mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) \geq \mu^* \left(\bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \mu^*(A_j).$$

Letting $n \to \infty$, we have

$$\mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) \geq \sum_{j=1}^{\infty} \mu^*(A_j).$$

Since, by subadditivity, we already know that

$$\mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) \leq \sum_{j=1}^{\infty} \mu^*(A_j),$$

we have shown that

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(A_j),$$

as required. $\square$

Exercise 11 Adapt the above proof to show that if $\{A_j\}_{j=1}^{\infty}$ is a countable collection of disjoint sets in $\mathcal{M}_0$ and if $X \subset \mathbb{R}$ is any set then

$$\mu^*\left(X \cap \bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(X \cap A_j).$$

$\mathcal{M}_0$ is a σ-algebra. Our aim is now to show that, in fact, $\mathcal{M}_0 = \mathcal{M}$, the σ-algebra of measurable sets (which, we recall, was defined to be the smallest σ-algebra containing both the Borel sets $\mathcal{B}$ and the null sets $\mathcal{N}$).
Our first step is to show that $\mathcal{M}_0$ is a σ-algebra.

Theorem A.10. *$\mathcal{M}_0$ is a σ-algebra.*

Proof. It is clear from the definition that $\emptyset \in \mathcal{M}_0$ and that if $A \in \mathcal{M}_0$ then $A^c \in \mathcal{M}_0$. It remains to show that $\mathcal{M}_0$ is closed under countable unions.
First we will show that if $A_1, A_2 \in \mathcal{M}_0$ then $A_1 \cup A_2 \in \mathcal{M}_0$. Take a test set $X \subset \mathbb{R}$; we need to show that

$$\mu^*((A_1 \cup A_2) \cap X) + \mu^*((A_1 \cup A_2)^c \cap X) = \mu^*(X).$$

Writing $X_1 = A_1 \backslash A_2 \cap X$, $X_2 = A_1 \cap A_2 \cap X$, $X_3 = A_2 \backslash A_1 \cap X$, and $X_4 = (A_1 \cup A_2)^c \cap X$, this is equivalent to

$$\mu^*(X_1 \cup X_2 \cup X_3) + \mu^*(X_4) = \mu^*(X).$$

Since $A_2 \in \mathcal{M}_0$, we may use $X_1 \cup X_2$ as a test set to obtain

$$\mu^*(X_1) + \mu^*(X_2) = \mu^*(X_1 \cup X_2) \tag{1}$$

and use $X_3 \cup X_4$ as a test set to obtain

$$\mu^*(X_3) + \mu^*(X_4) = \mu^*(X_3 \cup X_4). \tag{2}$$

Also, since $A_1 \in \mathcal{M}_0$ and using X as a test set we have

$$\mu^*(X_1 \cup X_2) + \mu^*(X_3 \cup X_4) = \mu^*(X). \tag{3}$$

Combining (1),(2) and (3) gives

$$\mu^*(X_1) + \mu^*(X_2) + \mu^*(X_3) + \mu^*(X_4) = \mu^*(X). \tag{4}$$

Now, since $A_1 \in \mathcal{M}_0$ and using $X_1 \cup X_2 \cup X_3$ as a test set, we have

$$\mu^*(X_1 \cup X_2) + \mu^*(X_3) = \mu^*(X_1 \cup X_2 \cup X_3).$$

By (1), we can rewrite this as

$$\mu^*(X_1) + \mu^*(X_2) + \mu^*(X_3) = \mu^*(X_1 \cup X_2 \cup X_3). \tag{5}$$

Combining (4) and (5) gives

$$\mu^*(X_1 \cup X_2 \cup X_3) + \mu^*(X_4) = \mu^*(X),$$

as required.
Using a simple induction argument, we can deduce (from the results for two sets) that $\mathcal{M}_0$ is closed under *finite* unions.
To complete the proof we have to consider countable unions. Suppose that $\{A_j\}_{j=1}^{\infty} \subset \mathcal{M}_0$. We can assume that these sets are disjoint by replacing A_2 by $A_2 \backslash A_1$, A_3 by $A_3 \backslash (A_1 \cup A_2)$ and, in general, A_j by $A_j \backslash \bigcup_{i=1}^{j-1} A_i$. (This procedure does not alter the union $\bigcup_{j=1}^{\infty} A_j$.) Let

$$B_n = A_1 \cup A_2 \cup \cdots \cup A_n,$$

then $B_n \in \mathcal{M}_0$. Furthermore, by Theorem A.9,

$$\mu^*(B_n) = \sum_{j=1}^{n} \mu^*(A_j).$$

By Exercise 11, for any test set X,

$$\begin{aligned}\mu^*(X) &= \mu^*(X \cap B_n) + \mu^*(X \cap B_n^c) \\ &= \sum_{j=1}^{n} \mu^*(X \cap A_j) + \mu^*(X \cap B_n^c).\end{aligned}$$

Write $A = \bigcup_{j=1}^{\infty} A_j$. Then $B_n \subset A$ for all $n \geq 1$, so $A^c \subset B_n^c$. Hence

$$\mu^*(X \cap B_n^c) \geq \mu^*(X \cap A^c)$$

and so

$$\mu^*(X) \geq \sum_{j=1}^{n} \mu^*(X \cap A_j) + \mu^*(X \cap A^c).$$

Letting $n \to \infty$ and again applying Exercise 11 gives

$$\begin{aligned}\mu^*(X) &\geq \sum_{j=1}^{\infty} \mu^*(X \cap A_j) + \mu^*(X \cap A^c) \\ &= \mu^*(X \cap A) + \mu^*(X \cap A^c).\end{aligned}$$

Therefore $A \in \mathcal{M}_0$. $\square$

Borel sets are in $\mathcal{M}_0$. We will prove that $\mathcal{B} \subset \mathcal{M}_0$ in two steps. The first is to show that semi-infinite intervals are in $\mathcal{M}_0$.

Lemma A.11. *For every $a \in \mathbb{R}$, the interval $(a, \infty) \in \mathcal{M}_0$.*

Proof. Let $X \subset \mathbb{R}$ be a test set and write $X_1 = X \cap (a, \infty)$, $X_2 = X \cap (-\infty, a]$. We need to show that

$$\mu^*(X_1) + \mu^*(X_2) \le \mu^*(X).$$

If $\mu^*(X) = +\infty$ then the result immediate, so assume that $\mu^*(X) < +\infty$. Then, by the definition of $\mu^*(X)$, given $\epsilon > 0$, there exists a cover $\{I_j\}$ of X by open intervals for which

$$\sum_j l(I_j) \le \mu^*(X) + \epsilon.$$

For each j, let $I'_j = I_j \cap (a, \infty)$, $I''_j = I_j \cap (-\infty, a]$. Then I'_j and I''_j are intervals (or empty) and

$$l(I_j) = l(I'_j) + l(I''_j) = \mu^*(I'_j) + \mu^*(I''_j).$$

Since $X_1 \subset \bigcup_j I'_j$, we have

$$\mu^*(X_1) \le \mu^* \left(\bigcup_j I'_j \right) \le \sum_j \mu^*(I'_j)$$

and since $X_2 \subset \bigcup_j I''_j$, we have

$$\mu^*(X_2) \le \mu^* \left(\bigcup_j I''_j \right) \le \sum_j \mu^*(I''_j).$$

Hence

$$\begin{aligned} \mu^*(X_1) + \mu^*(X_2) &\le \sum_j \left(\mu^*(I'_j) + \mu^*(I''_j) \right) \\ &= \sum_j l(I_j) \le \mu^*(X) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the required inequality holds. □

Theorem A.12. *We have $\mathcal{B} \subset \mathcal{M}_0$.*

Proof. We will repeatedly use the fact that $\mathcal{M}_0$ is a σ-algebra to show that $(a, \infty) \in \mathcal{M}_0$ for all a implies that all Borel sets are contained in $\mathcal{M}_0$.
By Lemma A.11, $(a, \infty) \in \mathcal{M}_0$, for every $a \in \mathbb{R}$. Since $\mathcal{M}_0$ is a σ-algebra, we also have $(a, \infty)^c = (-\infty, a] \in \mathcal{M}_0$. Since

$$(-\infty, b) = \bigcup_{n=1}^{\infty} \left(-\infty, b - \frac{1}{n} \right],$$

we have $(-\infty, b) \in \mathcal{M}_0$, for every $b \in \mathbb{R}$. Hence every open interval $(a, b) = (-\infty, b) \cap (a, \infty)$ is in $\mathcal{M}_0$. Since the Borel σ-algebra is the smallest σ-algebra containing all open intervals, we have $\mathcal{B} \subset \mathcal{M}_0$. □

$\mathcal{M}_0$ is equal to $\mathcal{M}$, the σ-algebra of measurable sets.

Proposition A.13. $\mathcal{M} \subset \mathcal{M}_0$

Proof. We have shown that $\mathcal{N} \subset \mathcal{M}_0$ and $\mathcal{B} \subset \mathcal{M}_0$, so $\mathcal{M}_0$ is a σ-algebra containing $\mathcal{N}$ and $\mathcal{B}$. Since $\mathcal{M}$ is the smallest σ-algebra with this property, we have $\mathcal{M} \subset \mathcal{M}_0$. $\square$

Lemma A.14. *Any bounded set $A \in \mathcal{M}_0$ can be written as $A = B \backslash N$, where $B \in \mathcal{B}$ and $N = A^c \cap B \in \mathcal{N}$.*

Proof. Let $A \in \mathcal{M}_0$ be bounded, i.e., $A \subset [-M, M]$, for some $M \geq 0$. Then $\mu^*(A) \leq \mu^*([-M, M]) = 2M < +\infty$.
By the regularity property of outer measure, for each $k \geq 1$ we can find an open set V_k such that $A \subset V_k$ and

$$\mu^*(V_k) \leq \mu^*(A) + \frac{1}{k}.$$

Let $B = \bigcap_{k=1}^{\infty} V_k \in \mathcal{B}$. Then, for each $k \geq 1$,

$$\mu^*(A) \leq \mu^*(B) \leq \mu^*(V_k) \leq \mu^*(A) + \frac{1}{k}$$

and so we see that $\mu^*(B) = \mu^*(A)$.
Now apply the definition of $\mathcal{M}_0$ to A with $X = B$. We get

$$\mu^*(A \cap B) + \mu^*(A^c \cap B) = \mu^*(B).$$

Since $A \subset B$, $A \cap B = A$ and this becomes

$$\mu^*(A) + \mu^*(A^c \cap B) = \mu^*(B).$$

Since $\mu^*(B) = \mu^*(A)$ and $\mu^*(A)$ is finite, we must have $\mu^*(A^c \cap B) = 0$. By Proposition A.2, $A^c \cap B$ is a null set. Thus we have found $B \in \mathcal{B}$ and $N = A^c \cap B \in \mathcal{N}$ such that $A = B \backslash N$. $\square$

Theorem A.15. *We have $\mathcal{M}_0 = \mathcal{M}$.*

Proof. For arbitrary $A \in \mathcal{M}_0$ and $m \in \mathbb{Z}$, write $A_m = A \cap [m, m+1]$. Then $A_m \in \mathcal{M}_0$ and A_m is bounded. By Lemma A.14, we can write $A_m = B \backslash N$ with $B \in \mathcal{B} \subset \mathcal{M}$ and $N \in \mathcal{N} \subset \mathcal{M}$. Since $\mathcal{M}_0$ is a σ-algebra, we get $A_m \in \mathcal{M}$ and hence

$$A = \bigcup_{m \in \mathbb{Z}} A_m \in \mathcal{M}.$$

Thus $\mathcal{M}_0 \subset \mathcal{M}$. Combined with Proposition A.13, this gives the result. $\square$

Definition We call the restriction of μ^* to $\mathcal{M}$ the Lebesgue measure on $\mathbb{R}$. We shall denote it by μ or, if we wish to be more precise, by $(\mathbb{R}, \mathcal{M}, \mu)$.

Remark You should note that the use of the symbol μ to denote Lebesgue measure is not universal.

Exercise 12 Prove that a measure which satisfies all the properties in Theorem 3.6 except (ii) (translation invariance) must also, in fact, satisfy (ii).

Exercise 13 **THIS EXERCISE IS NOT EXAMINABLE.** Prove that if ν is measure which satisfies properties (ii)-(vi) in Theorem 3.6 then there exists $c \in [0, +\infty)$ such that $\nu = c\mu$.

Exercise 14 Show that if $A \in \mathcal{M}$ with $\mu(A) = +\infty$ then, given any $\epsilon > 0$, there is an open set $V \subset \mathbb{R}$ such that $A \subset V$ and $\mu(V \backslash A) < \epsilon$. (*Hint:* recall that the case where $\mu(A) < +\infty$ was on Example Sheet 3. Try to deduce the required result from this.)

Remark We can also construct Lebesgue measure on $\mathbb{R}^n$, $n \geq 2$. In this setting, we are trying to generalize n-dimensional volume (area if $n = 2$, standard volume if $n = 3$). We shall denote n-dimensional Lebesgue measure by $\mu^{(n)}$. A set of the form

$$R = I_1 \times \cdots \times I_n,$$

where $I_1, \ldots, I_n$ are intervals is called an (n-dimensional) rectangle. The volume $\mathrm{vol}(R)$ of R is given by

$$\mathrm{vol}(R) = \prod_{j=1}^{n} l(I_j).$$

We wish to define $\mu^{(n)}$ so that $\mu^{(n)}(R) = \mathrm{vol}(R)$, for every rectangle R. Imitating our procedure for $n = 1$, we define an outer measure as follows:

$$\mu^{(n)*}(E) = \inf \left\{ \sum_j \mathrm{vol}(R_j) : \{R_j\} \text{ is a cover of } E \text{ by rectangles} \right\}.$$

It should be easy to convince yourself that $\mu^{(n)*}$ is an outer measure on $\mathbb{R}^n$. Essentially everything that we did for μ^* carries through for $\mu^{(n)*}$ except we have to work harder.

Part B: A Non-Measurable Set

So far we have not ruled out the possibility that *all* subsets of $\mathbb{R}$ are measurable. We now present an example of a non-measurable set. To do this we need to use:

Axiom of Choice. *Let $\mathcal{C}$ be any collection of non-empty sets. Then we can choose precisely one element from each of these sets. (More precisely, there is a function $f : \mathcal{C} \to \bigcup_{A\in\mathcal{C}} A$, such that $f(A) \in A$, for all $A \in \mathcal{C}$.)*

First, let us introduce some notation. For $x, y \in [0, 1)$ define

$$x \oplus y = x + y \pmod 1 = \begin{cases} x + y & \text{if } 0 \le x + y < 1 \\ x + y - 1 & \text{if } 1 \le x + y < 2, \end{cases}$$

and for $E \subset [0, 1)$, $x \in [0, 1)$ define

$$x \oplus E = \{x \oplus e : e \in E\} \subset [0, 1).$$

Lemma B.1. *If $E \subset [0, 1)$ is measurable and $x \in [0, 1)$ then $x \oplus E$ is measurable and $\mu(x \oplus E) = \mu(E)$.*

Proof. Let $E_1 = E \cap [0, 1 - x)$ and $E_2 = E \cap [1 - x, 1)$; then E_1 and E_2 are measurable (and disjoint) so

$$\mu(E) = \mu(E_1) + \mu(E_2).$$

It is easy to check that

$$x \oplus E_1 = x + E_1 \quad \text{and} \quad x \oplus E_2 = (x - 1) + E_2,$$

so that $x \oplus E_1$ and $x \oplus E_2$ are measurable and (by translation invariance)

$$\mu(x \oplus E_1) = \mu(E_1), \quad \mu(x \oplus E_2) = \mu(E_2).$$

It is also easy to check that $x \oplus E_1$ and $x \oplus E_2$ are disjoint and hence, since $x \oplus E = (x \oplus E_1) \cup (x \oplus E_2)$, $x \oplus E$ is measurable and

$$\mu(x \oplus E) = \mu(x \oplus E_1) + \mu(x \oplus E_2) = \mu(E_1) + \mu(E_2) = \mu(E). \qquad \square$$

Now define an *equivalence relation* $\sim$ on $[0, 1)$ by

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Q}.$$

Let $\mathcal{C}$ denote the collection of equivalence classes for this relation and apply the Axiom of Choice to obtain a set $E \subset [0, 1)$ consisting of one element from each equivalence class.

Theorem B.2. *E is not measurable.*

Proof. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals in $[0, 1)$. Let $x \in [0, 1)$; then $x \sim e$ for some $e \in E$, i.e., $x - e \in \mathbb{Q}$. Since $x - e \in (-1, 1)$, we have $x - e = r_n$ or $x - e = r_n - 1$ for some n. Rewriting this as $x = r_n + e$ or $x = r_n + e - 1$, shows that $x \in r_n \oplus E$.
Now suppose that $x \in (r_n \oplus E) \cap (r_m \oplus E)$. Then $x = r_n + e_n\ (-1) = r_m + e_m\ (-1)$, for some $e_n, e_m \in E$. (The -1 terms may or may not appear.) Hence $e_n - e_m \in \mathbb{Q}$, so that, since E contains one element from each equivalence class, $e_n = e_m$. Therefore $r_n = r_m$.
The above shows that

$$[0, 1) = \bigcup_{n=1}^{\infty} r_n \oplus E,$$

and that this union is disjoint. If E is measurable then so are the other sets in the union and

$$\begin{aligned} 1 = \mu([0, 1)) &= \mu\left(\bigcup_{n=1}^{\infty} r_n \oplus E\right) \\ &= \sum_{n=1}^{\infty} \mu(r_n \oplus E) \\ &= \sum_{n=1}^{\infty} \mu(E). \end{aligned}$$

But this sum is equal to either 0 or $+\infty$, depending on whether $\mu(E) = 0$ or $\mu(E) > 0$. Hence we have a contradiction and so E cannot be measurable. □

Theorem B.3. *There exist sets $A, B \subset \mathbb{R}$ such that $A \cap B = \emptyset$ but*

$$\mu^*(A) + \mu^*(B) \neq \mu^*(A \cup B).$$

Proof. Let E be the non-measurable set defined above. Since $\mathcal{M} = \mathcal{M}_0$, non-measurability means that we can find $X \subset \mathbb{R}$ such that

$$\mu^*(E \cap X) + \mu^*(E^c \cap X) \neq \mu^*(X).$$

Set $A = E \cap X$ and $B = E^c \cap X$. Then $A \cap B = \emptyset$ and $A \cup B = X$, so the above equation becomes

$$\mu^*(A) + \mu^*(B) \neq \mu^*(A \cup B),$$

as required. □

Part C: General Measures

So far we have studied very special measures on $\mathbb{R}$ and $\mathbb{R}^n$ – ones that that generalize length and volume, respectively. However, there is a general notion of a measure, which we will now introduce.

Let X be a set (no other structure is assumed) and let $\mathcal{B}$ be a σ-algebra of subsets of X. The pair $(X, \mathcal{B})$ is called a *measurable space.*

Definition Let $(X, \mathcal{B})$ be a measurable space. A function $m : \mathcal{B} \to \mathbb{R}^+ \cup \{+\infty\}$ satisfying

(i) $m(\emptyset) = 0$,

(ii) if $\{E_j\}_{j=1}^{\infty}$ is a countable collection of disjoint sets in $\mathcal{B}$ then

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j),$$

is called a *measure.* (If $m(X) < +\infty$ then we call m a finite measure.) The triple $(X, \mathcal{B}, m)$ is called a measure space.

Remarks

(i) Note that countable additivity implies monotonicity.

(ii) We do *not* assume that $\mathcal{B}$ is the largest σ-algebra on which we can define m as a measure. For example, if $\mathcal{M}$ denotes the Lebesgue measurable sets in $\mathbb{R}$ and $\mathcal{B}$ denotes the Borel sets in $\mathbb{R}$ then $(\mathbb{R}, \mathcal{M}, \mu)$ and $(\mathbb{R}, \mathcal{B}, \mu)$ are both measure spaces.

We can construct measures from outer measures in exactly the same way that we constructed Lebesgue measure from Lebesgue outer measure.

Definition Let X be a set. A map m^* from the set of all subsets of X to $\mathbb{R}^+ \cup \{+\infty\}$ is called an *outer measure* if

(i) $m^*(\emptyset) = 0$,

(ii) $E \subset F \implies m^*(E) \leq m^*(F)$,

(iii) if $\{E_j\}_{j=1}^{\infty}$ is a countable collection of sets in X then

$$m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*(E_j).$$

In this general context, we use the defining criterion of $\mathcal{M}_0$ as the definition of measurability:

Definition We say that $E \subset X$ is m^*-measurable (or just measurable, if no confusion arises) if, for every $A \subset X$,

$$m^*(E \cap A) + m^*(E^c \cap A) = m^*(A).$$

Write $\mathcal{M}$ for the collection of all m^*-measurable sets and let m denote the restriction of m^* to $\mathcal{M}$. We then have the following theorem.

Theorem C.1. *With the above definitions, $\mathcal{M}$ is a σ-algebra and $(X, \mathcal{M}, m)$ is a measure space.*

Proof. We've already proved it! If you look at the proofs of the corresponding results for Lebesgue measure you will see that all we used were the properties of outer measure and the definition of measurability. □

Example Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a non-negative measurable function. Then $m : \mathcal{M}(I) \to \mathbb{R}^+ \cup \{+\infty\}$ defined by

$$m(A) = \int_A f d\mu = \int_I f \chi_A \, d\mu \qquad (*)$$

is a measure.

Exercise 15 Show that m is a measure. Show that $m(I)$ is finite if and only if f is integrable over I.

Example For an arbitrary (non-empty) set X and for $x \in X$ define δ_x by

$$\delta_x(A) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ otherwise.} \end{cases}$$

Then $(X, \mathcal{P}, \delta_x)$ is a measure space, where $\mathcal{P}$ is the set of all subsets of X. (Such δ_x are called Dirac measures.)

Exercise 16 Show that $(X, \mathcal{P}, \delta_x)$ is a measure space. Show that any function $f : X \to \mathbb{R}$ is measurable with respect to δ_x and that

$$\int f \, d\delta_x = f(x).$$

Exercise 17 Let $I \subset \mathbb{R}$ be an interval and let $x \in I$. Show that there is no integrable function f such that for all $A \in \mathcal{M}(I)$ we have $\delta_x(A) = m(A)$, where m is defined by $(*)$.