MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Example Sheet 7

1. Give an example of an integrable function $f : [-\pi, \pi] \to \mathbb{R}^*$ which is *not* square integrable. (Hint: use a problem on Example Sheet 5.)

2. Prove Lemma 4.2: If $f, g \in L^2([-\pi, \pi], \mu, \mathbb{R})$ then

(i) **Hölder Inequality**:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |fg| \; d\mu \leq \|f\|_2 \|g\|_2,$$

with equality if and only if $|f| = c|g|$ μ-a.e. for some $c \in \mathbb{R}$, or $\|f\|_2 \|g\|_2 = 0$;

(ii) **Cauchy-Schwarz Inequality**:

$$\left| \frac{1}{\pi} \int_{-\pi}^{\pi} fg \; d\mu \right| \leq \|f\|_2 \|g\|_2,$$

with equality if and only if $f = cg$ μ-a.e. for some $c \in \mathbb{R}$, or $\|f\|_2 \|g\|_2 = 0$.

Hint: Use Lemma 4.1 or its corollary, for (i) with $\frac{f}{\|f\|_2}, \frac{g}{\|g\|_2}$, and for (ii) with tf, g in place of f, g (mainly to get the equality condition), where

$$t = \frac{1}{\|f\|_2^2} \frac{1}{\pi} \int_{-\pi}^{\pi} fg \, d\mu.$$

3. Show that if $f : [-\pi, \pi] \to \mathbb{R}$ is square integrable then f is integrable.

4. Show that, for $f \in L^2([-\pi, \pi], \mu, \mathbb{R})$ and for each $n \geq 1$,

$$\left(\int_{-\pi}^{\pi} |x^n f(x)| \, d\mu \right)^2 \leq \frac{2\pi^{2n+1}}{2n+1} \int_{-\pi}^{\pi} |f|^2 \, d\mu.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

5. Prove Corollary 4.6:
Given $f \in L^2([-\pi, \pi], \mu, \mathbb{R})$ and $\epsilon > 0$, we can find a continuous function $g : [-\pi, \pi] \to \mathbb{R}$ such that $g(-\pi) = g(\pi)$ and $\|f - g\|_2 < \epsilon$.

6. Let $f_n : [a, b] \to \mathbb{R}$, $n \geq 1$, be a sequence of measurable functions such that $|f_n(x)| < M$, for all $x \in [a, b]$. Suppose that f_n converges to f pointwise. Show that

$$\lim_{n \to +\infty} \int_a^b |f - f_n| \, d\mu = 0.$$

7. Let P be the countably infinite set

$$P = \{(p, q) : p, q \in \mathbb{N}, \ p < q\}.$$

For each $(p, q) \in P$, define a function $f_{(p,q)} : [0, 1] \to \mathbb{R}$ by

$$f_{(p,q)}(x) = \begin{cases} 1 & \text{if } x \in [\frac{p-1}{q}, \frac{p+1}{q}] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi : \mathbb{N} \to P$ be a bijection and define $g_n = f_{\phi(n)}$, for $n \in \mathbb{N}$.

(a) Show that

$$\lim_{n \to +\infty} \int_0^1 |g_n| \, d\mu = 0.$$

(b) Show that, for each $x \in [0, 1]$, there are infinitely many m such that $g_m(x) = 1$.

Part (b) means that there is no $x \in [0, 1]$ for which

$$\lim_{n \to +\infty} g_n(x) = 0.$$