MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Example Sheet 5

1. Prove Lemma 3.17:

If $f, g : [0,1] \to \mathbb{R}^*$ are non-negative measurable functions and $c \in \mathbb{R}^+$ then

(i)
$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu;$$

(ii)
$$\int cf\, d\mu = c \int f\, d\mu;$$

(iii) if $f \geq g$, then
$$\int f\, d\mu \geq \int g\, d\mu.$$

2. Prove Proposition 3.20:

If f, g are integrable and $c \in \mathbb{R}$ then cf and $f+g$ (if defined) are integrable and furthermore $\int f + g\ d\mu = \int f\ d\mu + \int g\ d\mu$ and $\int cf\ d\mu = c \int f\ d\mu$.

3. Suppose that $f, g : [0,1] \to \mathbb{R}$ are measurable functions such that $|f(x)| < M$ and $|g(x)| < M$ for all $x \in [0,1]$. Given $\epsilon > 0$, define $E = \{x \in [0,1] : |f(x) - g(x)| \geq \epsilon\}$. Show that
$$\int |f - g|\, d\mu < \epsilon + 2M\mu(E).$$

4. For each $p > 0$ define $f_p : [0,1] \to \mathbb{R}^*$ by

$$f_p(x) = \begin{cases} x^{-p} \text{ if } 0 < x \leq 1 \\ +\infty \text{ if } x = 0. \end{cases}$$

Show, using only the definition and properties we have proved, that f_p is integrable if and only if $p < 1$. (You may assume convergence properties of the series $\sum_{n=1}^{\infty} n^{-p}$.)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

5. Give an example of an integrable function $f : [0, 1] \to \mathbb{R}^*$ which has the value $+\infty$ at infinitely many points in $[0, 1]$.

6. (Hard) Let $f_n : [0, 1] \to \mathbb{R}$, $n \geq 1$, be a sequence of measurable functions converging pointwise to $f : [0, 1] \to \mathbb{R}$. Show that, for any $\epsilon > 0$, there is a set $A \subset [0, 1]$ with $\mu(A) < \epsilon$ such that f_n converges uniformly to f on $[0, 1]\backslash A$. (This result is called Egorov's Theorem.)
Hint: Consider the sets

$$E(n, m) = \bigcup_{k=n}^{\infty} \left\{ x \in [0, 1] : |f_k(x) - f(x)| \geq \frac{1}{m} \right\}.$$

For each m show that there exists n_m such that

$$\mu(E(n_m, m)) < \frac{\epsilon}{2^m}.$$

Then take $A = \bigcup_{m=1}^{\infty} E(n_m, m)$.