1. Prove Lemma 3.17:
If \( f, g : [0, 1] \to \mathbb{R}^* \) are non-negative measurable functions and \( c \in \mathbb{R}^+ \) then
   \begin{enumerate}[i)]
   \item \[ \int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu; \]
   \item \[ \int cf \, d\mu = c \int f \, d\mu; \]
   \item if \( f \geq g \), then \[ \int f \, d\mu \geq \int g \, d\mu. \]
   \end{enumerate}

2. Prove Proposition 3.20:
If \( f, g \) are integrable and \( c \in \mathbb{R} \) then \( cf \) and \( f + g \) (if defined) are integrable and furthermore \[ \int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu \] and \[ \int cf \, d\mu = c \int f \, d\mu. \]

3. Suppose that \( f, g : [0, 1] \to \mathbb{R} \) are measurable functions such that \( |f(x)| < M \) and \( |g(x)| < M \) for all \( x \in [0, 1] \). Given \( \epsilon > 0 \), define \( E = \{ x \in [0, 1] : |f(x) - g(x)| \geq \epsilon \} \). Show that \[ \int |f - g| \, d\mu < \epsilon + 2M \mu(E). \]

4. For each \( p > 0 \) define \( f_p : [0, 1] \to \mathbb{R}^* \) by
   \[ f_p(x) = \begin{cases} 
   x^{-p} & \text{if } 0 < x \leq 1 \\
   +\infty & \text{if } x = 0.
   \end{cases} \]

Show, using only the definition and properties we have proved, that \( f_p \) is integrable if and only if \( p < 1 \). (You may assume convergence properties of the series \( \sum_{n=1}^{\infty} n^{-p} \).)
5. Give an example of an integrable function $f : [0, 1] \to \mathbb{R}^+$ which has the value $+\infty$ at infinitely many points in $[0, 1]$.

6. (Hard) Let $f_n : [0, 1] \to \mathbb{R}$, $n \geq 1$, be a sequence of measurable functions converging pointwise to $f : [0, 1] \to \mathbb{R}$. Show that, for any $\epsilon > 0$, there is a set $A \subset [0, 1]$ with $\mu(A) < \epsilon$ such that $f_n$ converges uniformly to $f$ on $[0, 1] \setminus A$. (This result is called Egorov’s Theorem.)

*Hint:* Consider the sets

$$E(n, m) = \bigcup_{k=n}^{\infty} \left\{ x \in [0, 1] : |f_k(x) - f(x)| \geq \frac{1}{m} \right\}.$$

For each $m$ show that there exists $n_m$ such that

$$\mu(E(n_m, m)) < \frac{\epsilon}{2^m}.$$

Then take $A = \bigcup_{m=1}^{\infty} E(n_m, m)$. 