MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Example Sheet 6

1. Suppose that $f : \mathbb{R} \to \mathbb{R}^*$ is non-negative and measurable and, for $n \geq 1$, define

$$f_n(x) = \begin{cases} \min\{f(x), n\} & \text{for } -n \leq x \leq n \\ 0 & \text{for } |x| > n \end{cases}.$$

Show that

$$\lim_{n \to +\infty} \int_{\mathbb{R}} f_n \, d\mu = \int_{\mathbb{R}} f \, d\mu.$$

2. Let f_n, $n \geq 1$ be a sequence of non-negative measurable functions from $[0, 1]$ to $\mathbb{R}^*$. Show that

$$\int \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int f_n \, d\mu \right).$$

3. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{n\sqrt{x}}{1 + n^2 x^2}.$$

Let $G : [0, 1] \to \mathbb{R}$ be defined by

$$G(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $\lim_{n\to\infty} f_n = 0$ and that $f_n \leq G$ for all $n \geq 1$. Deduce that $\lim_{n\to\infty} \int f_n \, d\mu = 0$.

4. Show that $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

is an integrable function and that

$$\int f \, d\mu = \sum_{k=1}^{\infty} \frac{1}{(k + 1)k^2}.$$

(Hint: Use either the Monotone Convergence Theorem or the Dominated Convergence Theorem.)

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