MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Example Sheet 4

1. Prove Lemma 3.8:
If $f, g : [0,1] \to \mathbb{R}$ are simple functions and $k \in \mathbb{R}$ then

$$\text{(a) } kf, \quad \text{(b) } |f|, \quad \text{(c) } f+g, \quad \text{(d) } fg,$$

are all simple functions.

2. Prove Lemma 3.9:
If $f, g : [0,1] \to \mathbb{R}$ are simple functions and $a, b \in \mathbb{R}$ then

(i)

$$\int (af + bg)\, d\mu = a \int f\, d\mu + b \int g\, d\mu;$$

(ii) if $f(x) \le g(x)$ for all $x \in [0,1]$ then

$$\int f\, d\mu \le \int g\, d\mu;$$

(iii)

$$\left| \int f\, d\mu \right| \le \int |f|\, d\mu.$$

3. Suppose that f is a simple function and that

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

where the sets $E_i \in \mathcal{M}$ are not necessarily disjoint. Show that

$$\int f\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i).$$

(Hint: Use question 2.)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

4. Show that any constant function $f : [0,1] \to \mathbb{R}^*$ is measurable.

5. Prove Lemma 3.11:

(i) Simple functions are measurable.

(ii) Continuous functions are measurable.

(Hint: Use the following characterization of continuity: $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if, for every open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is an open set.)

6. Suppose that $f : [0,1] \to \mathbb{R}$ is increasing, i.e., if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$. Show that f is measurable.

7. Show that if $f : [0,1] \to \mathbb{R}^*$ is measurable then, for any $a \in \mathbb{R}^*$, the level set $\{x \in [0,1] : f(x) = a\}$ is in $\mathcal{M}([0,1])$.

8. For $f : [0,1] \to \mathbb{R}^*$ and $a \in \mathbb{R}$, we define f_a by

$$f_a(x) = \begin{cases} a & \text{if } f(x) \geq a \\ f(x) & \text{if } f(x) \leq a \end{cases}.$$

Show that if f is measurable then f_a is measurable.

9. [Hard.] Let $f : [0,1] \to \mathbb{R}$ be a simple function and let $\epsilon > 0$. Then there exists a step function $g : [0,1] \to \mathbb{R}$ and a set $A \subset [0,1]$ such that $\mu(A) < \epsilon$ and $f(x) = g(x)$, for all $x \notin A$.

(Hint: Use questions 6 and 8 on Sheet 3.)