MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Example Sheet 3

1. Prove Proposition 3.1:

Any countable set in $\mathbb{R}$ is a null set.

2. Imitating the construction of the Middle Third Cantor set, but removing middle fifths instead of middle thirds, construct the "Middle Fifth Cantor set" and show that is a null set.

3. (a) Show that $\mathcal{A} = \{A \subset [0,1] : A \text{ is countable or } [0,1]\backslash A \text{ is countable}\}$ is a σ-algebra of subsets of $[0,1]$.

(b) Show that $\mathcal{A} = \{A \subset [0,1] : A \text{ is a null set or } [0,1]\backslash A \text{ is a null set}\}$ is a σ-algebra of subsets of $[0,1]$.

4. Prove Lemma 3.4:

If $\{\mathcal{A}_\alpha\}_{\alpha\in\mathcal{I}}$ is an arbitrary collection of σ-algebras of subsets of a set X then

$$\bigcap_{\alpha\in\mathcal{I}} \mathcal{A}_\alpha$$

is also a σ-algebra.

5. Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{R}$ and let $X \in \mathcal{A}$. Let $\mathcal{A}(X)$ denote the collection of sets in $\mathcal{A}$ which are subsets of X. Show that $\mathcal{A}(X)$ is a σ-algebra of subsets of X.

6. Show that any open set in $\mathbb{R}$ may be written as a countable union of disjoint open intervals.

7. Prove the following:

(i) if $\{A_n\}_{n=1}^\infty$ is an increasing sequence of sets in $\mathcal{M}$ (i.e., $A_n \subset A_{n+1}$, for all $n \geq 1$) then

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to+\infty} \mu(A_n);$$

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(ii) if $\{A_n\}_{n=1}^{\infty}$ is an decreasing sequence of sets in $\mathcal{M}$ (i.e., $A_n \supset A_{n+1}$, for all $n \geq 1$) then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to+\infty} \mu(A_n),$$

provided $\mu(A_1) < +\infty$.

Show that (ii) does not hold without the assumption that $\mu(A_1) < +\infty$.

8. If $A \subset \mathbb{R}$ is a measurable set with $\mu(A) < +\infty$ then, given $\epsilon > 0$, there is an open set $U \supset A$ such that $\mu(U \backslash A) < \epsilon$. (*Hint:* use Theorem 3.6.)

9.(a) Let X denote the set of all irrational numbers in $[0, 1]$. Show that $\mu(X) = 1$.

(b) Prove that if Y is a closed subset of $[0, 1]$ and $\mu(Y) = 1$ then $Y = [0, 1]$.