MATH31011/MATH41011/MATH61011: FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Example Sheet 2

1. Prove Propositions 2.1:

(i) Let E be a countable set and let $f : E \to F$ be a surjection. Then F is countable.

(ii) Any subset of a countable set is countable.

2. Suppose that $x, y \in \mathbb{R}$ and

$$x = \sum_{n=1}^{\infty} a_n 10^{-n}, \quad y = \sum_{n=1}^{\infty} b_n 10^{-n},$$

with $a_n, b_n \in \{0, 1, 2, \ldots, 9\}$, for all $n \geq 1$. Show that if $x = y$ then one of the following holds

- For all $n \in \mathbb{N}$, $a_n = b_n$.
- There exists $m \in \mathbb{N}$ such that for all $n < m$, $a_n = b_n$, and $a_m = b_m + 1$, and for all $n > m$, $a_n = 0$ and $b_n = 9$.
- As the previous case with the roles of x and y swapped.

3. Show that if $X = \{0, 1\}$ then

$$X^{\mathbb{N}} = \{(x_n)_{n=1}^{\infty} : x_n \in X \text{ for all } n \in \mathbb{N}\}$$

is uncountable.

4.(a) Prove Propositions 2.10:

Let $E \subset \mathbb{R}$ be bounded above. Then

$$m = \sup E \quad \iff \quad \begin{cases} m \in \mathcal{U}(E), \text{ and} \\ \forall \epsilon > 0 \ \ \exists x \in E \text{ such that } m - \epsilon < x. \end{cases}$$

(b) Prove Propositions 2.11:

Let $E \subset \mathbb{R}$ be bounded below. Then

$$l = \inf E \quad \Longleftrightarrow \quad \begin{cases} l \in \mathcal{L}(E), \text{ and} \\ \forall \epsilon > 0 \ \ \exists x \in E \text{ such that } x < l + \epsilon. \end{cases}$$

5.(a) Prove Lemma 2.13:

For any sequence of real numbers x_n,

$$\limsup_{n \to +\infty} x_n = \lim_{n \to +\infty} \sup_{k \geq n} x_k.$$

(b) Prove Lemma 2.14:

For any sequence of real numbers x_n,

$$\liminf_{n \to +\infty} x_n = \lim_{n \to +\infty} \inf_{k \geq n} x_k.$$

6. Suppose that E and F are subsets of $\mathbb{R}$ which are bounded above. Show that

$$\sup(E \cup F) = \max\{\sup E, \sup F\}.$$