# MATH31011/MATH41011/MATH61011:
# Fourier Analysis and Lebesgue Integration

## Examples 1

**1.** (a) Show that, for $m, n \geq 1$,

$$\int_{-1}^{1} \cos\left(\frac{(2m-1)\pi x}{2}\right) \cos\left(\frac{(2n-1)\pi x}{2}\right) dx = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

(b) Deduce that, if

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{2}\right),$$

for $-1 < x < 1$, then

$$a_n = \int_{-1}^{1} f(x) \cos\left(\frac{(2n-1)\pi x}{2}\right) dx.$$

(You may assume that infinite sums and integrals may be interchanged where convenient.)

(c) Show that, if the function $f(x) = 1$ satisfies the assumption in (b), then the formula

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos\left(\frac{(2n-1)\pi x}{2}\right),$$

for $-1 < x < 1$ follows.

(d) By a judicious substitution, obtain a numerical value for the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

**2.** Let $f$ be the $2\pi$-periodic function satisfying

$$f(x) = \begin{cases} -1 & \text{if } -\pi \leq x \leq -\pi/2 \\ 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \pi/2 \leq x \leq \pi. \end{cases}$$

Show that the Fourier series for $f$ is equal to

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos((2n-1)x).$$

**3.** (a) Let $f_n : [a, b] \to \mathbb{R}$, $n \geq 1$, be a sequence of continuous functions. Suppose that $f_n$ converges *uniformly* to 0, as $n \to +\infty$. Show that the sequence of Riemann integrals satisfies

$$\lim_{n \to +\infty} \int_a^b f_n(x)\, dx = 0.$$

(You may assume basic properties of the integral.)

(b) Give an example of a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$ which converge *pointwise* to 0 but for which

$$\lim_{n \to +\infty} \int_a^b f_n(x)\, dx \neq 0.$$