MATH31011/MATH41011/MATH61011: FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Extra Reading: Solutions to Exercises 9-17

9.(a) Suppose that U is a bounded open subset of $\mathbb{R}$. Then U can be written as a countable union of (finite) pairwise disjoint open intervals:

$$U = \bigcup_i J_i.$$

Then, using the fact that $A \in \mathcal{S}$ and questions 7 and 8,

$$\begin{aligned}\mu^*(A \cap U) + \mu^*(A^c \cap U) &= \mu^*\left(A \cap \bigcup_i J_i\right) + \mu^*\left(A^c \cap \bigcup_i J_i\right)\\ &= \sum_i \left(\mu^*(A \cap J_i) + \mu^*(A^c \cap J_i)\right)\\ &= \sum_i \mu^*(J_i) = \mu^*\left(\bigcup_i J_i\right) = \mu^*(U),\end{aligned}$$

as required.
(b) Let X be a bounded set. First note that

$$\mu^*(X) \le \mu^*(A \cap X) + \mu^*(A^c \cap X)$$

follows from subadditivity.

By regularity, given $\epsilon > 0$, we can find a bounded open set U such that $X \subset U$ and

$$\mu^*(U) < \mu^*(X) + \epsilon.$$

Using the fact that $A \in \mathcal{S}$, monotonicity and part (a),

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) \le \mu^*(A \cap U) + \mu^*(A^c \cap U) = \mu^*(U) < \mu^*(X) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) \le \mu^*(X).$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Combining the two inequalities gives the result.

10. We need to show that $\mathcal{N} \subset \mathcal{M}_0$. Let $A \in \mathcal{N}$. Then, for any $X \subset \mathbb{R}$, $A \cap X \in \mathcal{N}$. Thus,

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) = \mu^*(A^c \cap X) \leq \mu^*(X).$$

This is enough to show that $A \in \mathcal{M}_0$.

11. Consider the first n sets $A_1, \ldots, A_n$. Set $A = A_1$ and $Y = X \cap (A_1 \cup A_2 \cup \cdots \cup A_n)$ in the definition of $\mathcal{M}_0$. Since $A_1 \in \mathcal{M}_0$, we obtain

$$\mu^*(X \cap A_1) + \mu^*(X \cap (A_2 \cup \cdots \cup A_n)) = \mu^*(X \cap (A_1 \cup \cdots \cup A_n)).$$

As in the proof of Theorem A.9, we repeat this argument a finite number of times to get

$$\mu^*\left(X \cap \bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu^*(X \cap A_j).$$

Since, for any $n \geq 1$, we have $\bigcup_{j=1}^{n} X \cap A_j \subset X \cap \bigcup_{j=1}^{\infty} A_j$, we obtain the inequality

$$\mu^*\left(X \cap \bigcup_{j=1}^{\infty} A_j\right) \geq \mu^*\left(X \cap \bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu^*(X \cap A_j).$$

Letting $n \to \infty$, we have

$$\mu^*\left(X \cap \bigcup_{j=1}^{\infty} A_j\right) \geq \sum_{j=1}^{\infty} \mu^*(X \cap A_j).$$

Since, by subadditivity, we already know that

$$\mu^*\left(X \cap \bigcup_{j=1}^{\infty} A_j\right) = \mu^*\left(\bigcup_{j=1}^{\infty} (X \cap A_j)\right) \leq \sum_{j=1}^{\infty} \mu^*(X \cap A_j),$$

we have shown that

$$\mu^*\left(X \cap \bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(X \cap A_j),$$

as required.

12. Suppose that ν satisfies properties (i) and (iii)-(vi) of Theorem 3.6. Let $E \in \mathcal{M}$. By (vi),

$$\nu(E) = \inf\{\nu(U) : U \text{ open}, \ E \subset U\}.$$

Each such U can be written as a countable disjoint union of open intervals:

$$U = \bigcup_i I_i.$$

By (i) and (iii),

$$\nu(U) = \sum_i \nu(I_i) = \sum_i l(I_i).$$

Now consider $x + E$. Sending U to $x + U$ gives a one-to-one correspondence between open sets containing E and open sets containing $x + E$. Thus,

$$\nu(x + E) = \inf\{\nu(x + U) : U \text{ open}, \ E \subset U\}.$$

Furthermore, $x+U$ can be written as a countable disjoint union of open intervals as follows:

$$x + U = \bigcup_i x + I_i.$$

Hence, using the basic fact that length is unchanged if an interval is translated,

$$\nu(x + U) = \sum_i \nu(x + I_i) = \sum_i l(x + I_i) = \sum_i l(I_i) = \nu(U).$$

Therefore,

$$\nu(x + E) = \inf\{\nu(U) : U \text{ open}, \ E \subset U\} = \nu(E).$$

13. (THIS EXERCISE IS NOT EXAMINABLE.) Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(x) = \nu([0, x]).$$

Then, using (iii) and (v),

$$\psi(x + y) = \nu([0, x + y]) = \nu([0, x]) + \nu([x, x + y]) - \nu(\{x\}) = \psi(x) + \nu([x, x + y)).$$

Using (ii),

$$\nu([x, x + y)) = \nu([0, y)) = \psi(y),$$

so

$$\psi(x + y) = \psi(x) + \psi(y).$$

Also, (iv) gives

$$x \leq x' \quad \Longrightarrow \quad \psi(x) \leq \psi(x').$$

Next, we claim that ψ is continuous. First, we note that, by (v),

$$\psi(0) = \nu(\{0\}) = 0.$$

By (vi),

$$0 = \nu(\{0\}) = \inf\{\nu(U) : U \text{ open}, \ 0 \in U\},$$

so, given $\epsilon > 0$, we can find an open set U_ϵ containing 0 such that $\nu(U_\epsilon) < \epsilon$. Since U_ϵ is open, we can find $x > 0$ sufficiently small that $[0, x] \subset U_\epsilon$, so that

$$\psi(x) = \nu([0, x]) \leq \nu(U_\epsilon) < \epsilon.$$

Since $\psi(x)$ decreases as x decreases, this shows that

$$\lim_{x \to 0+} \psi(x) = 0 = \psi(0),$$

i.e., ψ is continuous from above at 0.

Now take $y > 0$. We have

$$\lim_{z \to y+} \psi(z) = \lim_{x \to 0+} \psi(y + x) = \lim_{x \to 0+} (\psi(y) + \psi(x)) = \psi(y),$$

so ψ is continuous from above at y. One the other hand (provided $y - x \geq 0$),

$$\psi(y - x) = \psi(y) - \psi(x),$$

so

$$\lim_{z \to y-} \psi(z) = \lim_{x \to 0+} \psi(y - x) = \lim_{x \to 0+} (\psi(y) - \psi(x)) = \psi(y),$$

so ψ is continuous from below at y. Therefore, ψ is continuous from above at 0 and continuous at all other points.

We'll now show that ψ is linear. The property $\psi(x + y) = \psi((x) + \psi(y)$ implies that $\psi(nx) = n\psi(x)$ for any positive integer n. For positive integers p and q, we have

$$q\psi(px/q) = \psi(px) = p\psi(x),$$

which rearranges to

$$\psi(px/q) = p\psi(x)/q.$$

Now let $a \geq 0$ be arbitrary and let $r_n = p_n/q_n$ be a sequence of rationals with $\lim_{n \to +\infty} r_n = a$. Then, for any $x \geq 0$, $\lim_{n \to +\infty} r_n x = ax$, also. By continuity of ψ, we have

$$\psi(ax) = \lim_{n \to +\infty} \psi(r_n x) = \lim_{n \to +\infty} r_n \psi(x) = a\psi(x).$$

To summarise, we have shown that

(1) $\psi(x + y) = \psi(x) + \psi(y)$ for all $x, y \in \mathbb{R}^+$;
(2) $\psi(ax) = a\psi(x)$ for all $x \in \mathbb{R}^+$ and $a \in \mathbb{R}^+$,

i.e., ψ is a linear function. The only such functions take form $\psi(x) = cx$ and, since $\psi(x) \geq 0$, we must have $c \geq 0$.

Now, let $I = [a, b]$ be a closed interval. By (ii),

$$\nu(I) = \nu([0, b - a]) = \psi(b - a) = c(b - a) = cl(I).$$

Since

$$[a, b] = (a, b) \cup \{a\} \cup \{b\} = [a, b) \cup \{b\} = (a, b] \cup \{a\},$$

(v) tells us that

$$\nu(I) = cl(I)$$

for an arbitrary interval I.

Finally, we show that $\nu = c\mu$. Let $E \in \mathcal{M}$. By (vi), we have

$$\nu(E) = \inf\{\nu(U) : U \text{ open}, \ E \subset U\}.$$

Each such U can be written as a countable disjoint union of open intervals:

$$U = \bigcup_i I_i^U.$$

By (iii) and the result from the preceeding paragraph,

$$\nu(U) = \sum_i \nu(I_i^U) = \sum_i cl(I_i^U).$$

Thus,

$$\nu(E) = \inf\{\sum_i cl(I_i^U) : U \text{ open}, \ E \subset U\}.$$

On the other hand,

$$\mu(U) = \sum_i \mu(I_i^U) = \sum_i l(I_i^U).$$

Thus,

$$\mu(E) = \inf\{\mu(U) : U \text{ open}, \ E \subset U\} = \inf\{\sum_i l(I_i^U) : U \text{ open}, \ E \subset U\}.$$

This shows that $\nu = c\mu$, as required.

Remark for anyone interested. The additive property $\psi(x+y) = \psi(x)+\psi(y)$ is not enough to ensure that $\psi(x) = cx$. That there are additive but nonlinear functions follows from the Axiom of Choice:

Consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$. Then the set $\{1, \sqrt{2}\}$ consists of linearly independent elements. A consequence of the Axiom of Choice is that any linearly independent subset of a vector space can be extended to a basis. Let B be a basis extending

$\{1, \sqrt{2}\}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(1) = 1$ and $f(b) = 0$ for all $b \in B \backslash \{1\}$, and extend to $\mathbb{R}$ by linearity. Then f is linear as a function on $\mathbb{R}$ as a vector space over $\mathbb{Q}$, so, in particular,

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. However, it is *not* linear for $\mathbb{R}$ as a vector space over $\mathbb{R}$, for example

$$f(\sqrt{2}) = 0 \neq \sqrt{2} f(1).$$

14. Given $A \in \mathcal{M}$ with $\mu(A) = +\infty$, write

$$A_n = A \cap [n, n + 1).$$

Then $A_n \in \mathcal{M}$, $\mu(A_n) < +\infty$ and

$$A = \bigcup_{n \in \mathbb{Z}} A_n$$

with the union disjoint. By the corresponding problem on Example Sheet 3, given $\epsilon > 0$, for each n we can find an open set V_n such that $A_n \subset V_n$ and

$$\mu(V_n \backslash A_n) < \frac{\epsilon}{2^{|n|+2}}.$$

Now set $W = \bigcup_{n \in \mathbb{Z}} V_n$, an open set (as it is the countable union of open sets). Then we have

$$A = \bigcup_{n \in \mathbb{Z}} A_n \subset \bigcup_{n \in \mathbb{Z}} V_n = W$$

and

$$W \backslash A = \left(\bigcup_{n \in \mathbb{Z}} V_n \right) \backslash A = \bigcup_{n \in \mathbb{Z}} (V_n \backslash A) \subset \bigcup_{n \in \mathbb{Z}} (V_n \backslash A_n)$$

(since $A_n \subset A$). Finally by monotonicity and countable subbaditivity

$$\mu(W \backslash A) \leq \mu \left(\bigcup_{n \in \mathbb{Z}} (V_n \backslash A_n) \right) \leq \sum_{n \in \mathbb{Z}} \mu(V_n \backslash A_n) < \sum_{n \in \mathbb{Z}} \frac{\epsilon}{2^{|n|+2}} = \frac{\epsilon}{4} + 2 \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+2}} = \frac{3\epsilon}{4} < \epsilon.$$

15. Clearly $m(A) \geq 0$ for every $A \in \mathcal{M}(I)$. Furthermore, $\chi_{\emptyset} = 0$ so

$$m(\emptyset) = \int_{\emptyset} f \, d\mu = \int_I f \chi_{\emptyset} \, d\mu = 0.$$

If $E_1, E_2, \ldots \in \mathcal{M}(I)$ are disjoint then

$$\chi_{\bigcup_j E_j} = \sum_j \chi_{E_j}$$

and

$$m\left(\bigcup_j E_j\right) = \int_{\bigcup_j E_j} f\,d\mu = \int_I f\chi_{\bigcup_j E_j}\,d\mu$$
$$= \int_I f\left(\sum_j \chi_{E_j}\right)\,d\mu = \sum_j \int_I f\chi_{E_j}\,d\mu$$
$$= \sum_j \int_{E_j} f\,d\mu = \sum_j m(E_j).$$

We have that m is a finite measure when

$$m(I) = \int_I f\,d\mu = \int_I |f|\,d\mu$$

is finite. This is precisely the condition for f to be integrable (over I).

16. Trivially $\delta_x(A) \geq 0$ for every $A \subset X$ and $\delta_x(\emptyset) = 0$. Now let $E_1, E_2, \cdots$ be arbitrary disjoint subsets of X. Either $x \notin \bigcup_j E_j$ and $x \notin E_j$ for all j, or $x \in \bigcup_j E_j$ and $x \in E_j$ for exactly one value of j. In the former case,

$$\delta_x\left(\bigcup_j E_j\right) = 0 = \sum_j \delta_x(E_j).$$

In the latter case,

$$\delta_x\left(\bigcup_j E_j\right) = 1 = \sum_j \delta_x(E_j).$$

Thus, $(X, \mathcal{P}, \delta_x)$ is a measure space.

Let $f : X \to \mathbb{R}$ be an arbitrary function. For $c \in \mathbb{R}$,

$$\{x \in X \,:\, f(x) < c\} \in \mathcal{P},$$

so f is measurable.

Let $f = \sum_{i=1}^k \alpha_i \chi_{E_i}$ be a simple function, where $X = \bigcup_{i=1}^k E_i$ is a disjoint union. Then $x \in E_i$ for exactly one value of i. By definition,

$$\int f\,d\delta_x = \sum_{i=1}^k \alpha_i \delta_x(E_i) = \alpha_{i_0} = f(x),$$

where $x \in E_{i_0}$. Now let f_n be an increasing sequence of nonnegative simple functions converging pointwise to a (measurable) function f. Then

$$\int f\,d\delta_x = \lim_{n\to+\infty} \int f_n\,d\delta_x = \lim_{n\to+\infty} f_n(x) = f(x).$$

The extension to arbitrary functions is immediate.

17. Suppose that for all $A \in \mathcal{M}(I)$ we have

$$\delta_x(A) = \int_A f\, d\mu,$$

where $f : I \to \mathbb{R}$ is integrable. Note that $\{x\} \in \mathcal{M}(I)$ and $\chi_{\{x\}} = 0$ μ-a.e. Thus

$$1 = \delta_x(\{x\}) = \int_{\{x\}} f\, d\mu = \int_I f\chi_{\{x\}}\, d\mu = 0\,,$$

contraction. Therefore, δ_x does not have the claimed representation.