MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

EXTRA READING: SOLUTIONS TO EXERCISES 1-8

1.(i) Let

$$\mu_o^*(E) = \inf \left\{ \sum_j l(I_j) \, : \, \{I_j\} \text{ is a cover of } E \text{ by open intervals} \right\}.$$

First we note that the infimum in the definition of $\mu^*(E)$ is taken over a larger set than the infimum in the definition of $\mu_o^*(E)$. Therefore $\mu^*(E) \le \mu_o^*(E)$.

For the reverse inequality, take $\epsilon > 0$ and let $\{I_j\}_{j=1}^{\infty}$ be a cover of E by arbitrary intervals such that

$$\sum_{j=1}^{\infty} l(I_j) < \mu^*(E) + \frac{\epsilon}{2}.$$

If each I_j has endpoints $a_j \le b_j$ then setting

$$J_j = \left(a_j - \frac{\epsilon}{2^{j+2}}, b_j + \frac{\epsilon}{2^{j+2}} \right)$$

makes $\{J_j\}_{j=1}^{\infty}$ a cover of E by open intervals and

$$\sum_{j=1}^{\infty} l(J_j) = \sum_{j=1}^{\infty} \left(l(I_j) + \frac{\epsilon}{2^{j+1}} \right) = \sum_{j=1}^{\infty} l(I_j) + \frac{\epsilon}{2} < \mu^*(E) + \epsilon.$$

Since, by definition, $\mu_o^*(E) \le \sum_{j=1}^{\infty} l(J_j)$, this gives $\mu_o^*(E) < \mu^*(E) + \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $\mu_o^*(E) \le \mu^*(E)$, as required.

2. Clearly, for an interval I, $l(x+I) = l(I)$. There is a one-to-one correspondence between

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

covers for E and $x+E$, given by sending $\{I_j\}$ to $\{x+I_j\}$. Hence

$$\begin{aligned}
\mu^*(E) &= \inf\left\{\sum_j l(I_j) : \{I_j\} \text{ is a cover of } E \text{ by intervals}\right\} \\
&= \inf\left\{\sum_j l(I_j) : \{x+I_j\} \text{ is a cover of } x+E \text{ by intervals}\right\} \\
&= \inf\left\{\sum_j l(x+I_j) : \{x+I_j\} \text{ is a cover of } x+E \text{ by intervals}\right\} \\
&= \mu^*(x+E).
\end{aligned}$$

This completes the proof.

3. We want to show that a finite number of the J_α's cover $I=[a,b]$. We write

$$S = \{x \in [a,b] : [a,x] \text{ can be covered by a finite number of the } J_\alpha\}.$$

Since $a \in I$, there exists some $\alpha(a) \in \mathcal{A}$ such that $a \in J_{\alpha(a)}$. Since $J_{\alpha(a)}$ is open, there exists $\delta > 0$ such that $[a-\delta, a+\delta] \subset J_{\alpha(a)}$. In particular, $[a, a+\delta] \subset J_{\alpha(a)}$, so $a+\delta \in S$.

Hence S is non-empty and it is bounded above by b. Let $c = \sup S \leq b$. Since $a+\delta \in S$, we know that $a < c$. If $c < b$ then there exists some $\alpha(c) \in \mathcal{A}$ such that $[c-\epsilon, c+\epsilon] \subset J_{\alpha(c)}$, for some $\epsilon > 0$. By the definition of sup, there exists $x \in S$ with $c - \epsilon < x \leq c$. Then $[a,x]$ is covered by a finite number of the J_α's and $[c-\epsilon, c+\epsilon]$ is covered by one J_α, so

$$[a, c+\epsilon] = [a,x] \cup [c-\epsilon, c+\epsilon]$$

is covered by a finite number of the J_α's and hence $c+\epsilon \in S$. This contradicts the definition of c as $\sup S$. We conclude that $c = b$ (i.e., $b = \sup S$).

To complete the proof, we show that $b \in S$. Since $b \in I$, there exists $\alpha(b) \in \mathcal{A}$ such that $b \in J_{\alpha(b)}$. Since $J_{\alpha(b)}$ is open, there exists some $\eta > 0$ such that $[b-\eta, b+\eta] \subset J_{\alpha(b)}$. Since $b = \sup S$, there exists $y \in S$ such that $b - \eta < y \leq b$. Then $[a,y]$ is covered by a finite number of J_α's and $[b-\eta, b]$ is covered by one J_α, so

$$[a,b] = [a,y] \cup [b-\eta, b]$$

is covered by a finite number of J_α's, as required.

4. Let $A = \mathbb{Q} \cap [0,1]$. Then A is countable, so it is a null set and so $\mu^*(A) = 0$. But since $\mathbb{Q}$ is dense in $[0,1]$, if $A \subset \bigcup_{i=1}^n U_i$ then, taking closures,

$$[0,1] = \overline{A} \subset \bigcup_{i=1}^n \overline{U_i}.$$

We thus have

$$1 = \mu^*([0,1]) \leq \mu^*\left(\bigcup_{i=1}^{n} \overline{U_i}\right) \leq \sum_{i=1}^{n} l(\overline{U_i}) = \sum_{i=1}^{n} l(U_i).$$

5. If U is open and $A \subset U$ then, by monotonicity, $\mu^*(A) \leq \mu^*(U)$. Now let $\epsilon > 0$. By Exercise 1, $\mu^*(A) = \mu_o^*(A)$. By the definition of $\mu_o^*(A)$, there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} l(I_n) - \epsilon \leq \mu^*(A).$$

Write $O(\epsilon) = \bigcup_{n=1}^{\infty} I_n$. Clearly, $A \subset O(\epsilon)$ and

$$\mu^*(O(\epsilon)) \leq \sum_{n=1}^{\infty} l(I_n) \leq \mu^*(A) + \epsilon.$$

This shows that

$$\mu^*(A) = \inf\{\mu^*(U) : U \text{ is open and } A \subset U\}.$$

6.(a) Since $A \subset A \cup N$, monotonicity gives $\mu^*(A) \leq \mu^*(A \cup N)$. Subadditivity of μ^* gives

$$\mu^*(A \cup N) \leq \mu^*(A) + \mu^*(N) = \mu^*(A).$$

Therefore $\mu^*(A \cup N) = \mu^*(A)$, as required.
(b) Since $E \backslash F$ and $F \backslash E$ are subsets of $E \triangle F$, we have

$$\mu^*(E \backslash F) = 0 = \mu^*(F \backslash E).$$

Now

$$\begin{aligned} \mu^*(E) &= \mu^*((E \cap F) \cup E \backslash F) \\ &\leq \mu^*(E \cap F) + \mu^*(E \backslash F) \quad \text{(subadditivity)} \\ &= \mu^*(E \cap F) \leq \mu^*(F) \qquad \text{(montonicity)} . \end{aligned}$$

Similarly,

$$\begin{aligned} \mu^*(F) &= \mu^*((E \cap F) \cup F \backslash E) \\ &\leq \mu^*(E \cap F) + \mu^*(F \backslash E) \quad \text{(subadditivity)} \\ &= \mu^*(E \cap F) \leq \mu^*(E) \qquad \text{(montonicity)} . \end{aligned}$$

Hence $\mu^*(E) = \mu^*(F)$, as required.

7. By countable subadditivity,

$$\mu^*\left(\bigcup_j I_j\right) \leq \sum_j \mu^*(I_j) = \sum_j l(I_j).$$

(Alternatively, $\{I_j\}$ is a countable cover of $\bigcup_j I_j$ so, by the definition of μ^*, $\mu^*\left(\bigcup_j I_j\right) \leq \sum_j l(I_j)$.)

If $\mu^*\left(\bigcup_j I_j\right) = +\infty$, the above inequality is sufficient to prove the desired result. Suppose $\mu^*\left(\bigcup_j I_j\right) < +\infty$. We shall complete the proof in this case by obtaining the reverse inequality. Given $\epsilon > 0$, we can find a countable cover $\{J_k\}$ of $\bigcup_j I_j$ by open intervals such that

$$\sum_k l(J_k) < \mu^*\left(\bigcup_j I_j\right) + \epsilon.$$

Since the I_j are disjoint and open intervals, we may consider a new countable cover

$$\bigcup_j \bigcup_k \{J_k \cap I_j\}$$

of $\bigcup_j I_j$ by open intervals. Since the I_j are disjoint, there is no double counting and we have

$$\sum_j \sum_k l(J_k \cap I_j) \leq \sum_k l(J_k).$$

On the other hand,

$$\bigcup_j \bigcup_k \{J_k \cap I_j\}$$

is a collection of open intervals which covers the disjoint collection $\{I_j\}$, so

$$\sum_j l(I_j) \leq \sum_j \sum_k l(J_k \cap I_j).$$

Putting these together, we get

$$\sum_j l(I_j) < \mu^*\left(\bigcup_j I_j\right) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\sum_j l(I_j) \leq \mu^*\left(\bigcup_j I_j\right),$$

as required.

8. By countable subadditivity,

$$\mu^*\left(A \cap \bigcup_j I_j\right) = \mu^*\left(\bigcup_j (A \cap I_j)\right) \leq \sum_j \mu^*(A \cap I_j).$$

If $\mu^*\left(A \cap \bigcup_j I_j\right) = +\infty$, the above inequality is sufficient to prove the desired result. Suppose $\mu^*\left(A \cap \bigcup_j I_j\right) < +\infty$. We shall complete the proof in this case by obtaining the reverse inequality. Given $\epsilon > 0$, we can find a countable cover $\{J_k\}$ of $A \cap \bigcup_j I_j$ by open intervals such that

$$\sum_k l(J_k) < \mu^*\left(A \cap \bigcup_j I_j\right) + \epsilon.$$

Since the I_j are disjoint and open intervals, we may consider a new countable cover

$$\bigcup_j \bigcup_k \{J_k \cap I_j\}$$

of $A \cap \bigcup_j I_j$ by open intervals. Since the I_j are disjoint, there is no double counting and we have

$$\sum_j \sum_k l(J_k \cap I_j) \leq \sum_k l(J_k).$$

Now, for each j,

$$\bigcup_k \{J_k \cap I_j\}$$

is a collection of open intervals which covers the disjoint collection $A \cap I_j$. Thus

$$\mu^*(A \cap I_j) \leq \sum_k l(J_k \cap I_j).$$

Summing over j, we get

$$\sum_j \mu^*(A \cap I_j) \leq \sum_j \sum_k l(J_k \cap I_j).$$

Combining inequalities,

$$\sum_j \mu^*(A \cap I_j) < \mu^*\left(A \cap \bigcup_j I_j\right) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\sum_j \mu^*(A \cap I_j) \leq \mu^*\left(A \cap \bigcup_j I_j\right),$$

as required.