MATH31011/MATH41011/MATH61011:
FOURIER ANALYSIS AND LEBESGUE INTEGRATION

Appendix to Chapter 4: Continuous functions are dense in $L^2([\pi,\pi],\mu,\mathbb{R})$.

THIS MATERIAL IS NOT EXAMINABLE.

Theorem 4.5 *Continuous functions are $\|\cdot\|_2$-dense in $L^2([-\pi,\pi],\mu,\mathbb{R})$. In other words, given $f \in L^2([-\pi,\pi],\mu,\mathbb{R})$ and $\epsilon > 0$, we can find a continuous function $g : [-\pi,\pi] \to \mathbb{R}$ such that $\|f-g\|_2 < \epsilon$.*

To prove the theorem we will need the following lemmas:

Lemma A *Let $f : [-\pi,\pi] \to \mathbb{R}$ be a simple function and let $\epsilon > 0$. Then there exists a step function $g : [-\pi,\pi] \to \mathbb{R}$ and a set $A \subset [-\pi,\pi]$ such that $\mu(A) < \epsilon$ and $f(x) = g(x)$ for all $x \notin A$.*

Proof This is as Exercise 9 on Example Sheet 4. □

Lemma B *Let $f : [-\pi,\pi] \to \mathbb{R}$ be a simple function and let $\epsilon > 0$. Then there exists a continuous function $g : [-\pi,\pi] \to \mathbb{R}$ and a set $A \subset [-\pi,\pi]$ such that $\mu(A) < \epsilon$ and $f(x) = g(x)$ for all $x \notin A$.*

Proof Let $f : [-\pi,\pi] \to \mathbb{R}$ be a simple function and let $\epsilon > 0$ be given. By Lemma A, we can find a step function $h : [-\pi,\pi] \to \mathbb{R}$ and a set $A_0 \subset [-\pi,\pi]$ such that $\mu(A_0) < \frac{\epsilon}{2}$ and $f(x) = h(x)$ for all $x \notin A_0$. We can write

$$h = \sum_{j=1}^{m} \beta_j \chi_{I_j} ,$$

where $I_1, \ldots, I_m$ are disjoint (non-trivial) intervals (which may be open or closed or half closed) such that $\bigcup_{j=1}^m I_j = [-\pi,\pi]$. There are $m-1$ points, $y_1, \ldots, y_{m-1}$ say, where these intervals meet. Let

$$\eta \le \frac{\epsilon}{4(m-1)}, \quad \eta \le \min\left\{\frac{l(I_j)}{2} : j = 1, \ldots, m\right\}$$

We define a continuous function as follows: On $[y_j - \eta, y_j + \eta]$ define g to be the linear function satisfying $g(y_j - \eta) = h(y_j - \eta)$ and $g(y_j + \eta) = h(y_j + \eta)$ and for the remaining x (that is, $x \in [-\pi,\pi] \setminus \bigcup_{j=1}^{m-1}[y_j - \eta, y_j + \eta]$), define $g(x) = h(x)$. Now define

$$A = A_0 \cup \bigcup_{j=1}^{m-1} [y_j - \eta, y_j + \eta] .$$

Then $g(x) = f(x)$ for all $x \notin A$ and since

$$\mu(A) \le \mu(A_0) + 2\eta(m-1) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

the lemma is proved. □

Proof of Theorem 4.5 Let $f : [-\pi, \pi] \to \mathbb{R}$ be a square integrable function and let $\epsilon > 0$ be given.

Let f^+ be the positive part of f. We have proved that we can find an increasing sequence of nonnegative simple functions h_n^+ such that $h_n^+ \to f^+$ pointwise, as $n \to +\infty$. Thus $(f^+ - h_n^+)^2 \to 0$ pointwise as $n \to +\infty$ and

$$|(f^+ - h_n^+)^2| \leq |f^+|^2 + 2|f^+||h_n^+| + |h_n^+|^2 \leq 4|f^+|^2$$

which is integrable. Applying the Dominated Convergence Theorem,

$$\|f^+ - h_n^+\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |f^+ - h_n^+|^2 \, d\mu\right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to +\infty\,.$$

Now let f^- denote the negative part of f. Arguing in a similar way, we can find an increasing sequence of simple functions h_n^- such that $h_n^- \to f^-$ pointwise, as $n \to +\infty$ and

$$\|f^- - h_n^-\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |f^- - h_n^-|^2 \, d\mu\right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to +\infty\,.$$

Set $h_n = h_n^+ - h_n^-$. Then h_n are simple and

$$\|f - h_n\|_2 \leq \|f^+ - h_n^+\|_2 + \|f^- - h_n^-\|_2 \to 0 \quad \text{as} \quad n \to +\infty\,.$$

In particular, we can choose N sufficiently large so that $\|f - h_N\|_2 < \frac{\epsilon}{2}$. Let $\max_{x\in[-\pi,\pi]} |h_N(x)| = M$.

By Lemma B we can find a continuous function $g : [-\pi, \pi] \to \mathbb{R}$ and a set $A \subset [-\pi, \pi]$ such that

$$\mu(A) < \frac{\pi\epsilon^2}{16M^2}$$

and $g(x) = h_N(x)$ for $x \notin A$. Furthermore, it is clear from the construction of g that we may assume $|g| \leq \max_{x\in[-\pi,\pi]} |h_N(x)| \leq M$, and hence

$$|h_N - g|^2 \leq |h_N|^2 + 2|h_N||g| + |g|^2 \leq 4|h_N|^2 \leq 4M^2\,.$$

Thus

$$\|h_N - g\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |h_N - g|^2 \chi_A \, d\mu \leq \frac{1}{\pi} 4M^2 \mu(A) < \frac{\epsilon^2}{4}$$

so that $\|h_N - g\|_2 < \frac{\epsilon}{4}$.

Finally, we have the estimate

$$\|f - g\|_2 \leq \|f - h_N\|_2 + \|h_N - g\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as required.