Solutions 9

13.10 Exercise

For $T = Th(\langle \mathbb{Z}; \dot{+}; 0, 1\rangle)$ prove there are uncountably many 1-types (over T)

Solution For any integer $m \geq 2$, let $m|v_1$ be the formula

$$\exists v_2 \big(\underbrace{\cdots((v_2 \dot{+} v_2) \dot{+} v_2) \dot{+} \cdots v_2}_{m\ v_2\text{'s}}) \doteq v_1 \big)$$

Thus $\mathfrak{G} \models m|v_1[a]$ (for $\mathfrak{G} = \langle \mathbb{Z}; +; 0, 1\rangle$, $a \in \mathbb{Z}$) if and only if m divides a.

Now for each set $S \subseteq Pr.$, where $Pr. = \{2, 3, 5, 7, 11, \ldots\}$ denotes the set of all prime numbers, let q_S' be the following set of formulas:

$$q_S' := \{ m|v_1 : m \in S\} \cup \{\neg m|v_1 : m \in Pr. \setminus S\}.$$

Now let $q_S := \left\{ \left(\bigwedge_{i=1}^{r} \phi_i \right) : r \geq 1,\ \phi_1, \ldots, \phi_r \in q_S' \right\}$.

Then q_S is closed under conjunction. Also, if $\phi \in q_S$ then $T \models \exists v_1 \phi$. For if ϕ is $\left(\bigwedge_{i=1}^{r} \phi_i\right)$ and

ϕ_i is either $m_i|v_1$ (for $m_i \in S$) or $\neg m_i|v_1$ (for $m_i \in Pr \setminus S$)

then let N be the product of those m_i's of the first kind. Then $\mathfrak{G} \models \phi[N]$, so $\mathfrak{G} \models \exists v_1 \phi$, so $T \models \exists v_1 \phi$.

Thus q_S is a partial 1-type (over T). Let p_S be a 1-type (over T) such that $q_S \subseteq p_S$ (see 13.3 (1)).

Claim: if $S \neq S'$ ($S \subseteq Pr$, $S' \subseteq Pr$), then $p_S \neq p_{S'}$.

Proof: ~~For~~ suppose $m \in S$, $m \notin S'$ (say). Then $m|v_1 \in q_S$ and $\neg m|v_1 \in q_{S'}$. So $m|v_1 \in p_S$ and $\neg m|v_1 \in p_{S'}$. So not $m|v_1 \in p_{S'}$ and hence $p_S \neq p_{S'}$.

$\square$ claim.

So we have an injection from the set of all subsets of Pr into the set of 1-types (over T). So the latter set is uncountable.

13.11 Exercise

Let T be ω-categorical and $\mathfrak{A} \models T$. For $S \subseteq A$, $\langle S \rangle$ denotes the smallest substructure of $\mathfrak{A}$ containing S. Prove that if S is finite then so is $\langle S \rangle$.

Solution

Say $S = \{a_1, \ldots, a_n\}$. Let $\mathcal{T}_n$ be the collection of terms of $\mathcal{L}$ (:= the language for T) with variables amongst $v_1, \ldots, v_n$. Then one easily shows that
$\langle S \rangle = \{ \tau^{\mathfrak{A}}[a_1, \ldots, a_n] : \tau \in \mathcal{T}_n \}$ --- $(*)$

Now for each $\tau \in \mathcal{T}_n$, let $\phi_\tau \in F_{n+1}(\mathcal{L})$ be the formula

$$\tau(v_1, \ldots, v_n) \doteq v_{n+1}.$$

Since $F_{n+1}(\mathcal{L}) / E_{n+1}(T)$ is finite it follows that there are $\tau_1, \ldots, \tau_N \in \mathcal{T}_n$ such that for any $\tau \in \mathcal{T}_n$ there is some $i \in \{1, \ldots, N\}$ such that $\phi_\tau \, E_{n+1}(T) \, \phi_{\tau_i}$.

We show that $\langle S \rangle = \{ \tau_1^{\mathfrak{A}}[a_1, \ldots, a_n], \ldots, \tau_N^{\mathfrak{A}}[a_1, \ldots, a_n] \}$.

For suppose $b \in \langle S \rangle$. Then by $(*)$, $b = \tau^{\mathfrak{A}}[a_1, \ldots, a_n]$ for some $\tau \in \mathcal{T}_n$. Choose i s.t. $\phi_\tau \, E_{n+1}(T) \, \phi_{\tau_i}$.

Then $T \models \forall v_1, \ldots, \forall v_{n+1} (\phi_\tau \Leftrightarrow \phi_{\tau_i})$ so

$$\mathfrak{A} \models \forall v_1 \cdots \forall v_{n+1} \left(\tau(v_1, \ldots, v_n) \doteq v_{n+1} \Leftrightarrow \tau_i(v_1, \ldots, v_n) \doteq v_{n+1} \right)$$

In particular, choose a_j for v_j and b for v_{n+1}, we have

$$\mathfrak{A} \models \left(\tau(v_1, \ldots, v_n) \doteq v_{n+1} \Leftrightarrow \tau_i(v_1, \ldots, v_n) \doteq v_{n+1} \right)[a_1, \ldots, a_n, b],$$

i.e. $\tau^{\mathfrak{A}}[a_1, \ldots, a_n] = b \Leftrightarrow \tau_i^{\mathfrak{A}}[a_1, \ldots, a_n] = b$.

But the LHS is true, so $\tau_i^{\mathfrak{A}}[a_1, \ldots, a_n] = b$, which shows that $b \in \{ \tau_1^{\mathfrak{A}}[a_1, \ldots, a_n], \ldots, \tau_N^{\mathfrak{A}}[a_1, \ldots, a_n] \}$, as required.