Solutions 8

12.4 Exercise

If Σ is any complete $\mathcal{L}$-theory and ϕ is any $\mathcal{L}$-sentence, then if $\mathfrak{A} \models \phi$ for *some* model $\mathfrak{A} \models \Sigma$, then $\mathfrak{A} \models \phi$ for *every* model $\mathfrak{A} \models \Sigma$, and hence $\Sigma \models \phi$.

Solution

Suppose there exists $\mathfrak{B} \models \Sigma$ such that $\mathfrak{B} \models \neg\phi$. Since there is some $\mathfrak{A} \models \Sigma$ such that $\mathfrak{A} \models \phi$, it follows that *neither* $\Sigma \models \phi$ *nor* $\Sigma \models \neg\phi$, so Σ is not complete.

12.5 Exercise

Suppose that $\mathcal{L}$ is a finite, purely relational language and that T is a complete $\mathcal{L}$-theory without finite models. Suppose that for all $n \geq 1$, each $E_n(T)$-equivalence class contains a QF formula. Prove that T is ω-categorical.

Solution

Let $n \geq 1$. We must show (by 12.2) that $F_n(\mathcal{L})/E_n(T)$ is finite, and by our assumption it is sufficient to show that $Q_n(\mathcal{L})/E_n(T)$ is finite where $Q_n(\mathcal{L})$ denotes the set of quantifier-free formulas in $F_n(\mathcal{L})$.

Now every atomic $\mathcal{L}$-formula of $F_n(\mathcal{L})$ has the form

$$P(v_{i_1}, \ldots, v_{i_r})$$

where P is a relation symbol of $\mathcal{L}$ (which is r-ary in this case), and $1 \leq i_1, \ldots, i_r \leq n$. Clearly there are only finitely many $\mathcal{L}$-words of this form (since there are only finitely many P's, and n is fixed).

By the disjunctive normal form theorem (of the propositional calculus) every QF formula is logically equivalent to one of the form

$$\bigvee_{i=1}^{k} \left(\bigwedge_{j=1}^{l_i} \phi_{i,j} \right) \quad \text{----} \quad (*)$$

where each $\phi_{i,j}$ is either atomic or negated atomic. Now up to logical equivalence, we may assume that for each i, the $\mathcal{L}$-words $\phi_{i,1}, \ldots, \phi_{i,l_i}$ are all distinct. So there are only finitely many $\mathcal{L}$-words of the form

$$\bigwedge_{j=1}^{l_i} \phi_{i,j} .$$

Clearly we may also suppose that for $i \neq i'$ the $\mathcal{L}$-words $\bigwedge_{j=1}^{l_i} \phi_{i,j}$ and $\bigwedge_{j=1}^{l_{i'}} \phi_{i',j}$ are different, and hence there are only finitely many $\mathcal{L}$-words of this form.

So there are only finitely many possibilities for expressions of the form $(*)$. Thus $\mathcal{Q}_n(\mathcal{L})/E_n(T)$ is finite for each n. (Though $|\mathcal{Q}_n(\mathcal{L})/E_n(T)|$ would appear to increase very fast with n if one does the calculations here.)

12.6 Exercise

Let $\mathfrak{A} = \langle \mathbb{N}; <; +, \cdot; 0, 1\rangle$. Use the Ryll-Nardzewski theorem to prove that there exists a structure $\mathfrak{B}$ such that $\mathfrak{B} \equiv \mathfrak{A}$ but $\mathfrak{B} \not\cong \mathfrak{A}$.

Solution

Let $T = Th(\mathfrak{A})$. Then T is certainly a complete theory and has no finite models (since $\chi_n \in T$ for all $n \geq 1$). Now, for each $i \geq 1$, let $\phi_i \in F_1(\mathcal{L})$ be the formula

$$(\underbrace{\dot{1} \dot{+} \dot{1} \dot{+} \cdots \dot{+} \dot{1}}_{i\text{-times}}) \doteq v_1 .$$

(I put a dot over a symbol to make it the corresponding symbol of the language.)

Now if T were ω-categorical, then $F_1(\mathcal{L})/_{E_1(T)}$ would be finite. In particular there would exist $i \neq j$ such that $T \models \forall v_1 (\phi_i \leftrightarrow \phi_j)$. So $\mathfrak{A} \models \forall v_1 (\phi_i \Leftrightarrow \phi_j)$.

In particular $\mathfrak{A} \models (\phi_i \Leftrightarrow \phi_j)[i]$. However, $\mathfrak{A} \models \phi_i[i]$ but not $\mathfrak{A} \models \phi_j[i]$. This contradiction shows that T is not ω-categorical, and hence there exist countable $\mathfrak{A}_1, \mathfrak{A}_2 \models T$ such that $\mathfrak{A}_1 \not\cong \mathfrak{A}_2$. So at least one of these, say $\mathfrak{A}_1$, is not isomorphic to $\mathfrak{A}$. But $\mathfrak{A}_1 \equiv \mathfrak{A}$ since T is complete.

13.5 Exercises

(1) Suppose $\mathfrak{A} \models T$ and that $\langle a_1, \ldots, a_n\rangle \in A^n$ realises the principal n-type p. Then if ϕ is a principal formula for p, ϕ is also principal for $\langle a_1, \ldots, a_n\rangle$ (in $\mathfrak{A}$).

<u>Solution</u>

Since $\langle a_1, \ldots, a_n \rangle$ realises p we have that $\mathfrak{A} \models \psi[a_1, \ldots, a_n]$ for all $\psi \in p$. Also if $\psi \notin p$ then $\neg\psi \in p$ so $\mathfrak{A} \models \neg\psi[a_1, \ldots, a_n]$. Thus for any $\psi \in F_n(\mathcal{L})$,
$\mathfrak{A} \models \psi[a_1, \ldots, a_n] \iff \psi \in p$. ...... $(*)$
In particular, $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$. Further, if $\psi \in F_n(\mathcal{L})$ is any formula such that $\mathfrak{A} \models \psi[a_1, \ldots, a_n]$ then $\psi \in p$ (by $(*)$) and so $T \models \forall v_1 \cdots \forall v_n (\phi \to \psi)$ (since ϕ is principal for p).

Thus ϕ is principal for $\langle a_1, \ldots, a_n \rangle$ in $\mathfrak{A}$, as required.

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(2) Suppose that ϕ, ψ are both principal for the n-type p. Then $\phi\, E_n(T)\, \psi$.

<u>Solution</u>

Since ϕ is principal for p and $\psi \in p$ we have $T \models \forall v_1 \cdots \forall v_n (\phi \to \psi)$. Similarly $T \models \forall v_1 \cdots v_n (\psi \to \phi)$. Thus $T \models \forall v_1 \cdots v_n (\phi \leftrightarrow \psi)$, i.e. $\phi\, E_n(T)\, \psi$.
