Solutions 7

11.9 Exercise

(1) Prove the 0-1 law for sentences of the language containing just n constant symbols $c_1, \ldots, c_n$.

Solution

Let $\mathcal{L}$ be this language. Since each constant symbol may be interpreted as any element of $\{1, \ldots, N\}$, the number, $\Theta(\mathcal{L}; N)$, of $\mathcal{L}$-structures with this domain is N^n. Let ϕ be the sentence

$$\bigwedge_{1 \le i \ne j \le n} \neg c_i \doteq c_j$$

(or just $c_1 \doteq c_1$ if $n = 1$).

Then $\Theta(\phi; N) = N \cdot (N-1) \cdots (N-n+1)$.

So $p(\phi; N) = \dfrac{\Theta(\phi; N)}{\Theta(\mathcal{L}; N)} = 1 \cdot \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{(n-1)}{N}\right)$

$\rightarrow 1 \cdot (1-0) \cdots (1-0) = 1$

as $N \to \infty$ (as n is fixed).

Thus $\phi \in T^{as}(\mathcal{L})$. Also $\chi_d \in T^{as}(\mathcal{L})$ for each $d \geq 2$, so if $\langle A; e_1, \ldots, e_n \rangle \models T^{as}(\mathcal{L})$ and $\langle B; d_1, \ldots, d_n \rangle \models T^{as}(\mathcal{L})$ with A, B countable, then mapping each e_i to d_i $(i = 1, \ldots, n)$ and $A \setminus \{e_1, \ldots, e_n\}$ to $B \setminus \{d_1, \ldots, d_n\}$ bijectively (which is possible, as they are both countably infinite sets) clearly defines an isomorphism from $\langle A; e_1, \ldots, e_n \rangle$ to $\langle B; d_1, \ldots, d_n \rangle$. Thus

$T^{as}(\mathcal{L})$ is ω-categorical and hence, by Vaught's test, is complete. The required result now follows from 11.5.

11.9 <u>Exercise</u>

(2) Let E_2 be the sentence saying that "P is an equivalence relation with just two equivalence classes. Prove that $p_{E_2}(\phi; N) \to 1$ or $\to 0$ as $N \to \infty$.

<u>Solution</u>

E_2 is the sentence

$$((((\forall v_1\ P(v_1,v_1) \wedge \forall v_1 \forall v_2\ (P(v_1,v_2) \to P(v_2,v_1))) \wedge$$
$$\forall v_1 \forall v_2 \forall v_3\ ((P(v_1,v_2) \wedge P(v_2,v_3)) \to P(v_1,v_3))) \wedge$$
$$\exists v_1 \exists v_2 (\neg P(v_1,v_2) \wedge \forall v_3\ (P(v_1,v_3) \vee P(v_2,v_3))))\ )$$

If $\langle \{1,\ldots,N\}; R \rangle \models E_2$ then we have two non-empty equivalence classes. There are $2^N - 2$ possibilities for an equivalence class, so

$\theta(E_2; N) = \frac{1}{2}(2^N - 2) = 2^{N-1} - 1$.

It seems reasonable to guess that each equivalence class gets arbitrarily large almost surely. So let Γ_n be the sentence (for $n \geq 2$):

$$\exists v_1 \cdots \exists v_n \exists w_1 \cdots \exists w_n\ (((\bigwedge_{\substack{i=1\\j=1}}^{n} P(v_i,v_j) \wedge \bigwedge_{1\leq i\neq j\leq n} \neg v_i \doteq v_j) \wedge$$
$$(\bigwedge_{\substack{i=1\\j=1}}^{n} P(w_i,w_j) \wedge \bigwedge_{1\leq i\neq j\leq n} \neg w_i \doteq w_j)) \wedge \neg P(v_1,w_1))$$

Now fix $n \geq 2$. The number of models $\langle \{1, .., N\}, R \rangle$ of E_2 that fail to satisfy Γ_n (for $N \geq 2n$) is at most the number of non-empty subsets of $\{1, .., N\}$ of size at most $n-1$. This is

$$\sum_{k=1}^{n-1} \binom{N}{k},$$

which is bounded above by $n \cdot N^n$.

Thus $$\frac{\theta_{E_2}(\neg\Gamma_n ; N)}{\theta(E_2 ; N)} \leq \frac{n \cdot N^n}{2^{N-1}-1} \to 0 \text{ as } N \to \infty.$$

Thus, as usual, $$\frac{\theta_{E_2}(\Gamma_n ; N)}{\theta(E_2 ; N)} \to 1 \text{ as } N \to \infty,$$

so $\Gamma_n \in T^{as}(E_2)$. Now to establish the required result it is sufficient to prove that $T^{as}(E_2)$ is complete for which, since $\{\Gamma_n : n \geq 2\}$ has no finite models, it is sufficient to show that $E_2 \cup \{\Gamma_n : n \geq 2\}$ is ω-categorical (for then $T^{as}(E_2)$ is too).

But a countably infinite model of $E_2 \cup \{\Gamma_n : n \geq 2\}$ is just an equivalence relation with two countably infinite equivalence classes, and any two such structures are clearly isomorphic by just "matching up" the equivalence classes.
