10.6 Corollary (of the Compactness Theorem)

Let \( \Sigma \) be a set of \( L \)-sentences, and \( \phi \) any \( L \)-sentence. If \( \Sigma \models \phi \) then there exists a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( \Sigma_0 \models \phi \).

Proof: If \( \Sigma \models \phi \) then \( \Sigma \cup \{ \neg \phi \} \) has no model. By the (contrapositive of) the Compactness Theorem, there exists \( \Sigma_0 \subseteq \Sigma \) such that \( \Sigma_0 \cup \{ \neg \phi \} \) has no model. But then every model of \( \Sigma_0 \) is necessarily a model of \( \phi \), so \( \Sigma_0 \models \phi \).

11.1 Exercise

Evaluate \( \Theta(L; N) \) for \( L = L_0 \) when \( \sigma = <I, J, K, \rho, \mu> \) and \( I, J, K \) are finite.

Solution

For \( i \in I \), there are \( 2^{N^{\rho(i)}} \) subsets of \( \{1, \ldots, N^3\}^{\rho(i)} \).

For \( j \in J \), there are \( N^{N^{\rho(j)}} \) functions from \( \{1, \ldots, N^3\}^{\rho(j)} \) to \( \{1, \ldots, N^3\} \).

For \( k \in K \), there are \( N \) elements of \( \{1, \ldots, N^3\} \).

Thus \( \Theta(L; N) = |\Theta(L; N)| = \prod_{i \in I} 2^{N^{\rho(i)}} \cdot \prod_{j \in J} N^{N^{\rho(j)}} \cdot N^{\mu_1} \),

where an empty product is defined to be 1.

11.4 Exercise

Prove that for any \( \phi_1, \ldots, \phi_n \in T^0(L) \), \( (i_1, \ldots, i_n) \in T^0(L) \).

Solution

Let \( \varepsilon > 0 \). Then \( \varepsilon \rho_1 > 0 \), so there exists \( N_0 > 0 \)
such that $\forall n \geq N_0$, and for each $i = 1, \ldots, r$

$$|1 - p(\phi_i^c; N)| < \frac{\varepsilon}{r + 1} \tag{*}$$

But

$$\Theta(- (\bigwedge_{i=1}^r \phi_i^c); N) = \bigcup_{i=1}^r \Theta(- \phi_i^c; N),$$

so

$$\Theta(- (\bigwedge_{i=1}^r \phi_i^c); N) \leq \sum_{i=1}^r \Theta(- \phi_i^c; N)$$

$$= \sum_{i=1}^r \left(\Theta(\phi_i^c; N) - \Theta(\phi_i^c; N^1)\right)$$

$\therefore$

$$\theta(- (\bigwedge_{i=1}^r \phi_i^c); N) \leq \sum_{i=1}^r (1 - p(\phi_i^c; N)) < \frac{r \varepsilon}{r + 1} \ (by \ (*)).$$

$\therefore$

$$1 \geq p((- (\bigwedge_{i=1}^r \phi_i^c); N) = 1 - p(- (\bigwedge_{i=1}^r \phi_i^c); N) > 1 - \varepsilon \ \text{for} \ N \geq N_0.$$}

Thus

$$p((- (\bigwedge_{i=1}^r \phi_i^c); N) \to 1 \ \text{as} \ N \to \infty \ \text{and hence} \ (\bigwedge_{i=1}^r \phi_i^c) \in \mathbb{T}^a(L).$$

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**Easy exercise (p.51)**

$\Gamma_0$ has no finite models.

**Solution**

Actually, the graph with 2 points and one edge (i.e., $\{1, 2, 3, 4, 5, 6\}$) satisfies $\Gamma_0$, so we should add $K_3$ to $\Gamma_0$.

Now suppose $\langle A; R \rangle \models \Gamma_0$ with $A = \{a_1, \ldots, a_N\}$ ($N \geq 3$).

If $\langle a_i, a_j \rangle \in R$ for all $i, j$ with $i \neq j$, choose $w_1 = a_1, w_2 = a_2, v_1 = v_2 = a_3$.

Then for every $c \in A$, either $\langle a_1, c \rangle \in R$ or $\langle a_2, c \rangle \in R$, so $\Pi_2$ is false in $\langle A; R \rangle$ - contradiction.

So suppose, say, $\langle a_1, a_2 \rangle \in R$. Set $v_1 = a_2, v_2 = a_3, \ldots, v_n = a_{n+1}$ and $w_1, \ldots, w_n$ all equal to $a_1$. Then there is no $c \in A$ such that $\langle c, a_j \rangle$ for $j = 1, \ldots, n+1$, since we cannot have $c = a_1$. But then $c = a_j$ for some $j = 2, \ldots, n+1$, and then $\langle c, a_j \rangle \in R$ (as $R$ is irreflexive).

Hence $\langle A; R \rangle \not\models \Pi_2$ - contradiction.