Solutions 6

10.6 **Corollary** (of the Compactness Theorem)

Let Σ be a set of $\mathcal{L}$-sentences, and ϕ any $\mathcal{L}$-sentence. If $\Sigma \models \phi$ then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \phi$.

Proof.

If $\Sigma \models \phi$ then $\Sigma \cup \{\neg\phi\}$ has no model. By the (contrapositive of) the Compactness Theorem, there exists $\Sigma_0 \subseteq_{fin} \Sigma$ such that $\Sigma_0 \cup \{\neg\phi\}$ has no model. But then every model of Σ_0 is necessarily a model of ϕ, so $\Sigma_0 \models \phi$.

$\square$

11.1 **Exercise**

Evaluate $\theta(\mathcal{L};N)$ for $\mathcal{L} = \mathcal{L}_\sigma$ where $\sigma = \langle I, J, K, \rho, \mu \rangle$ and I, J, K are finite

Solution

For $i \in I$ there are $2^{N^{\rho(i)}}$ subsets of $\{1,\dots,N\}^{\rho(i)}$.

For $j \in J$ there are $N^{N^{\mu(j)}}$ functions from $\{1,\dots,N\}^{\mu(j)}$ to $\{1,\dots,N\}$.

For $k \in K$, there are N elements of $\{1,\dots,N\}$.

Thus
$$\theta(\mathcal{L};N) = |\mathcal{S}(\mathcal{L};N)| = \prod_{i\in I} 2^{N^{\rho(i)}} \cdot \prod_{j \in J} N^{N^{\mu(j)}} \cdot N^{|K|},$$

where an empty product is defined to be 1.

11.4 **Exercise**

Prove that for any $\phi_1,\dots,\phi_r \in T^{as}(\mathcal{L})$, $(\bigwedge_{i=1}^{r} \phi_i) \in T^{as}(\mathcal{L})$.

Solution

Let $\varepsilon > 0$. Then $\varepsilon/_{r+1} > 0$, so there exists $N_0 > 0$

such that $\forall N \geq N_0$, and for each $i = 1, \ldots, r$

$$|1 - p(\phi_i; N)| < \frac{\varepsilon}{r+1} \quad \cdots (*).$$

But $$\mathcal{S}(\neg(\bigwedge_{i=1}^{r} \phi_i); N) = \bigcup_{i=1}^{r} \mathcal{S}(\neg \phi_i; N), \text{ so}$$

$$\theta(\neg(\bigwedge_{i=1}^{r} \phi_i); N) \leq \sum_{i=1}^{r} \theta(\neg\phi_i; N)$$

$$= \sum_{i=1}^{r} (\theta(\mathcal{L}; N) - \theta(\phi_i; N))$$

$$\therefore \quad p(\neg(\bigwedge_{i=1}^{r} \phi_i); N) \leq \sum_{i=1}^{r} (1 - p(\phi_i; N)) < \frac{r\varepsilon}{r+1} \quad (\text{by } (*)).$$

$$\therefore \quad 1 \geq p((\bigwedge_{i=1}^{r} \phi_i); N) = 1 - p(\neg(\bigwedge_{i=1}^{r} \phi_i); N) > 1 - \varepsilon \quad \text{for } N \geq N_0.$$

Thus $p((\bigwedge_{i=1}^{r} \phi_i); N) \to 1$ as $N \to \infty$ and hence $(\bigwedge_{i=1}^{r} \phi_i) \in T^{as}(\mathcal{L})$.

<u>Easy exercise (p51)</u>

Γ_∞ has no finite models.

<u>Solution</u>

Actually, the graph with 2 points and one edge (i.e. $\langle \{1,2\}; \{\langle 1,2\rangle, \langle 2,1\rangle\}\rangle$) satisfies Γ_∞, so we should add χ_3 to Γ_∞.

Now suppose $\langle A; R\rangle \models \Gamma_\infty$ with $A = \{a_1, \ldots, a_N\}$ $(N \geq 3)$.

If $\langle a_i, a_j\rangle \in R$ for all i, j with $i \neq j$, choose $w_1 = a_1$, $w_2 = a_2$, $v_1 = v_2 = a_3$. Then for every $c \in A$, either $\langle a_1, c\rangle \in R$ or $\langle a_2, c\rangle \in R$, so Γ_2 is false in $\langle A; R\rangle$ – contradiction.

So suppose, say, $\langle a_1, a_2\rangle \notin R$. Set $v_1 = a_2, v_2 = a_3, \ldots, v_n = a_{n+1}$ and $w_1, \ldots, w_n$ all equal to a_1. Then there is no $c \in A$ such that $\langle c, a_j\rangle$ for $j = 2, \ldots, n+1$, since we cannot have $c = a_1$. But then $c = a_j$ for some $j = 2, \ldots, n+1$, and then $\langle c, a_j\rangle \notin R$ (as R is irreflexive).

Hence $\langle A; R\rangle \not\models \Gamma_n$ – contradiction.