Solutions 5

8.2.1 Exercise

Let σ be a countable similarity type. Prove that there are only countably many $\mathcal{L}_\sigma$-formulas. (This is essentially what 8.2.1 on page 33 of the notes is asking.)

Solution

Let S be the collection of all (logical and nonlogical) basic symbols of $\mathcal{L}_\sigma$. Then S is a finite union of countable sets, and hence countable. Now for each n, the collection $\mathcal{F}_n$ of $\mathcal{L}_\sigma$-formulas of length n may be regarded as a subset of S^n $(= \underbrace{S \times S \times \cdots \times S}_{n})$, and hence is countable. Thus the collection $\bigcup_{n \geq 1} \mathcal{F}_n$ of all $\mathcal{L}_\sigma$-formulas, being a countable union of countable sets, is countable.

$\square$

9.4 Exercise

Let T be a set of $\mathcal{L}_\sigma$-sentences. Then T is satisfiable if and only if there is no $\mathcal{L}_\sigma$-sentence ϕ such that $T \models \phi$ and $T \models \neg\phi$.

Solution

$\Rightarrow$: Say $\mathfrak{A} \models T$. If $\mathfrak{A} \models \phi$, then it is not the case that $T \models \neg\phi$. If not $\mathfrak{A} \models \phi$, then $\mathfrak{A} \models \neg\phi$ so it is not the case that $T \models \phi$.

$\Leftarrow$: Suppose $T \models \phi$ and $T \models \neg\phi$. If $\mathfrak{A} \models T$, then both $\mathfrak{A} \models \phi$ and $\mathfrak{A} \models \neg\phi$ which is absurd. So no such $\mathfrak{A}$ exists.

9.14.1 Exercise

Prove that DLO has no finite models.

Solution

If $\mathfrak{A}$ is a model of the first three (LO) axioms, then clearly $\mathfrak{A}$ is isomorphic to $\langle \{1, \ldots, N\}, < \rangle$ for some $N \geq 1$, and where $<$ is the usual order on $\{1, \ldots, N\}$. But then $\mathfrak{A}$ does not satisfy the fourth (DENSENESS) axiom (nor the fifth, nor the sixth).

9.14.2 Exercise

Assuming DLO is complete, prove that $\langle \mathbb{Q}; < \rangle \equiv \langle \mathbb{R}; < \rangle$. Use the automorphism test to show that more is true, namely that $\langle \mathbb{Q}; < \rangle \preccurlyeq \langle \mathbb{R}; < \rangle$.

Solution

Let ϕ be a sentence (of the language of orderings) such that $\langle \mathbb{Q}; < \rangle \models \phi$. Then since $\langle \mathbb{Q}; < \rangle \models \text{DLO}$, we cannot have $\text{DLO} \models \neg\phi$. So by the completeness of DLO we have $\text{DLO} \models \phi$. But $\langle \mathbb{R}; < \rangle \models \text{DLO}$, so $\langle \mathbb{R}; < \rangle \models \phi$. The converse implication follows by considering $\neg\phi$. Thus $\langle \mathbb{Q}; < \rangle \equiv \langle \mathbb{R}; < \rangle$.

To show $\langle \mathbb{Q}; < \rangle \preccurlyeq \langle \mathbb{R}; < \rangle$ by the automorphism test, let X be a finite subset of $\mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$. Say $X = \{q_1, \ldots, q_n\}$ in strictly increasing order. We choose $d \in \mathbb{Q}$ to have the same relative order as b, e.g. if $b < q_1$, set $d = q_1 - 1$; if $q_i < b < q_{i+1}$ (some $i = 1, \ldots, n-1$) set $d = \frac{q_i + q_{i+1}}{2}$; if $b > q_n$ set $d = q_n + 1$.

Now write $X \cup \{d\}$ $(\subseteq \mathbb{Q})$ in increasing order as $r_1 < r_2 < \cdots < r_{n+1}$, and $X \cup \{b\}$ as $s_1 < s_2 < \cdots < s_{n+1}$. Notice that for some i_0, $r_{i_0} = d$ and $s_{i_0} = b$.

So we are looking for a strictly increasing, surjective function $\pi : \mathbb{R} \to \mathbb{R}$ such that $\pi(s_i) = r_i$ for $i = 1, \ldots, n+1$. For then π will be an automorphism of $\langle \mathbb{R}; < \rangle$ satisfying $\pi(x) = x$ for each $x \in X$ (because $s_i = r_i \in X$ for $i \neq i_0$) and $\pi(s_{i_0}) = r_{i_0}$, i.e. $\pi(b) = d \in \mathbb{Q}$, as required by the automorphism test.

Now π may be easily constructed as a piecewise linear function:

$$\pi(x) = \begin{cases} x + (r_1 - s_1) & \text{if } x \leq s_1 ; \\ (x(r_{i+1} - r_i) + r_i s_{i+1} - s_i r_{i+1})(s_{i+1} - s_i)^{-1}, & \text{if } s_i < x \leq s_{i+1} ; \\ x + r_{n+1} - s_{n+1}, & \text{if } x > s_{n+1} . \end{cases}$$

9.15 <u>Exercise</u>

Let $\mathcal{L}_\sigma$ contain just one unary function symbol F. Let T be T_∞ together with the sentence

$$\forall v_1 \; (F(F(v_1)) \doteq v_1 \wedge \neg F(v_1) \doteq v_1).$$

Prove that T is (satisfiable and) complete.

<u>Solution</u>

Since $T_\infty \subseteq T$, T has no finite models. Also, it is easy to see that any model of T consists of infinitely many "2-cycles" :—

$\langle A; f \rangle$: 

If two such structures are countable, they are clearly isomorphic by matching up the 2-cycles. So T is ω-categorical, hence complete by Vaught's Test.
