Solutions 4

6.4 Exercise

Deduce from the first part of 6.3 that if $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ and ϕ is an existential formula, then ϕ is π-preserved from $\mathfrak{A}$ to $\mathfrak{B}$.

Solution

We are given that ϕ has the form

$$\exists v_{i_1} \exists v_{i_2} \cdots \exists v_{i_r} \psi$$

where ψ is a QF formula and $i_1, \ldots, i_r \geq 1$.

Suppose that $\bar{a} \in A^\omega$ and that $\mathfrak{A} \models \phi[\bar{a}]$. Then by repeated use of 5.1(5) it follows that there exist $b_1, \ldots, b_r \in A$ such that $\mathfrak{A} \models \psi[\bar{a}(i_1/b_1)(i_2/b_2)\cdots(i_r/b_r)]$.
By 6.3 we have $\mathfrak{B} \models \psi[\pi(\bar{a}(i_1/b_1)\cdots(i_r/b_r))]$.
So (letting $d_j = \pi(b_j)$ for $j=1,\ldots,r$), it follows that there are $d_1, \ldots, d_r \in B$ such that $\mathfrak{B} \models \psi[\pi(\bar{a})(i_1/d_1)\cdots(i_r/d_r)]$.
(Note that $\pi(\bar{a}(i_1/b_1)\cdots(i_r/b_r)) = \pi(\bar{a})(i_1/d_1)\cdots(i_r/d_r)$.)
So by 5.1(5) again, we get $\mathfrak{B} \models \exists v_{i_1} \cdots \exists v_{i_r} \psi[\pi(\bar{a})]$, i.e. $\mathfrak{B} \models \phi[\pi(\bar{a})]$, as required.

6.5 Exercise

Let $\mathfrak{A} = \langle \mathbb{Z}; +; 0 \rangle$. Prove that there is an existential formula E such that for all $\bar{a} \in \mathbb{Z}^\omega$, $\mathfrak{A} \models E[\bar{a}]$ iff a_1 is even, but there is no such existential formula if we replace "even" by "odd" here.

Solution

If F_1 is the binary function symbol of the appropriate language, then we take E to be

$$\exists v_2 \ F_1(v_2, v_2) \doteq v_1 .$$

Then for $\bar{a} \in \mathbb{Z}^\omega$, $\mathfrak{A} \models E[\bar{a}]$ iff for some $b \in \mathbb{Z}$, $\mathfrak{A} \models F_1(v_2,v_2) \doteq v_1 \ [\bar{a}(2/b)]$ iff for some $b \in \mathbb{Z}$, $b+b = a_1$ iff a_1 is even.

Clearly E is existential as $F_1(v_2, v_2) \doteq v_1$ is a QF (in fact atomic) formula.

Now suppose Θ were an existential formula such that $\mathfrak{A} \models \Theta[\bar{a}]$ iff a_1 is odd (for all $\bar{a} \in \mathbb{Z}^\omega$).

Define $\pi : \mathbb{Z} \to \mathbb{Z}$ by $\pi(x) = 2\cdot x$ (for $\in \mathbb{Z}$).

Then $\pi(x+y) = 2\cdot(x+y) = 2\cdot x + 2\cdot y = \pi(x) + \pi(y)$ (for $x, y \in \mathbb{Z}$) and $\pi(0) = 2\cdot 0 = 0$. So $\pi : \mathfrak{A} \hookrightarrow \mathfrak{A}$.

Let $\bar{a} = \langle 1, 1, 1, \ldots \rangle$. Then $a_1 = 1$ is odd, so $\mathfrak{A} \models \Theta[\bar{a}]$. Since Θ is existential we have, by 6.4, that $\mathfrak{A} \models \Theta[\pi(\bar{a})]$. So $\pi(a_1)$ is odd, i.e. $\pi(1) = 2$ is odd! Contradiction. So there is no such formula Θ.

7.7 Exercise

With $\mathcal{L}$ the empty language and A a finite set, which sets B satisfy $\langle A; \rangle \preccurlyeq \langle B; \rangle$?

Solution

Only $B = A$ satisfies $\langle A; \rangle \preccurlyeq \langle B; A \rangle$. For obviously $\langle A; \rangle \preccurlyeq \langle A; \rangle$.

Now suppose B is any set such that $\langle A; \rangle \preccurlyeq \langle B; \rangle$. Then $A \subseteq B$. Suppose $\operatorname{card}(A) = n$, say $A = \{a_1, \ldots, a_n\}$. Then, if $\bar{a} = \langle a_1, a_2, \ldots, a_n, a_n, a_n, \ldots \rangle$ we have $\mathfrak{A} \models \forall v_{n+1} \left(\bigvee_{i=1}^{n} v_{n+1} \doteq v_i \right) [\bar{a}]$. Since $\mathfrak{A} \preccurlyeq \mathfrak{B}$, it follows that $\mathfrak{B} \models \forall v_{n+1} \left(\bigvee_{i=1}^{n} v_{n+1} \doteq v_i \right) [\bar{a}]$. Thus for all $b \in B$, $b = a_1$ or $b = a_2$ or $\cdots$ or $b = a_n$. So $A = B$.

7.8 <u>Exercise</u>

Prove that $\langle \mathbb{R}; +; 0 \rangle \preccurlyeq \langle \mathbb{C}; +; 0 \rangle$.

<u>Solution</u>

Obviously $\langle \mathbb{R}; +; 0 \rangle \subseteq \langle \mathbb{C}; +; 0 \rangle$. We now apply the automorphism test (7.5). So let $X \subseteq_{\text{fin}} \mathbb{R}$.

Let $V(X)$ denote the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ generated by X. I.e. if $X = \{x_1, \ldots, x_n\}$, then

$$V(X) := \{q_1 x_1 + q_2 x_2 + \cdots + q_n x_n : q_1, \ldots, q_n \in \mathbb{Q}\}.$$

Let $\{y_1, \ldots, y_r\}$ be a basis for $V(X)$ (where $r \leq n$).
Since $V(X)$ is countable there exists some $d \in \mathbb{R} \setminus V(X)$.
Then $\{y_1, \ldots, y_r, d\}$ is linearly independent.
Now suppose $b \in \mathbb{C} \setminus \mathbb{R}$. Since $y_1, \ldots, y_r, d$ are all real – and hence so is any $\mathbb{Q}$-linear combination of $y_1, \ldots, y_r, d$ – and b has non-zero imaginary part, it follows that $\{y_1, \ldots, y_r, d, b\}$ is a $\mathbb{Q}$-linearly independent set.
So there exists $S \subseteq \mathbb{C}$ such that $\{y_1, \ldots, y_r, d, b\} \cup S =: T$ is a basis for $\mathbb{C}$ (as a $\mathbb{Q}$-vector space).
Now define the permutation π' of this set T by

$$\pi'(z) = \begin{cases} d & \text{if } z = b \\ b & \text{if } z = d \\ z & \text{if } z \in \{y_1, \ldots, y_r\} \cup S. \end{cases}$$

Now extend π' to all of $\mathbb{C}$ by linearity :–

$$\pi(q_1 t_1 + \cdots + q_N t_N) = q_1 \pi'(t_1) + \cdots + q_N \pi'(t_N).$$

(for all $t_1, \ldots, t_N \in T$, $q_1, \ldots, q_N \in \mathbb{Q}$).

By the theory of vector spaces this is a linear bijection from $\mathbb{C}$ to $\mathbb{C}$. In particular, it is an automorphism of $\langle \mathbb{C}; +; 0 \rangle$. Further $\pi(b) = d \in \mathbb{R}$ and π is the identity on the chosen basis for $V(X)$. So it is the identity on $V(X)$ and, in particular, $\pi(x_i) = x_i$ for $i = 1, \ldots, n$.
So we have verified the hypotheses of 7.5, so $\langle \mathbb{R}; +; 0 \rangle \preccurlyeq \langle \mathbb{C}; +; 0 \rangle$.
